Supplementary material for Simultaneous modeling of choice, confidence, and response time in visual perception

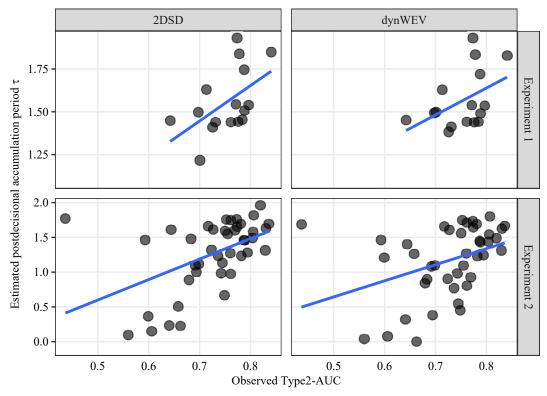
Preprint Version

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1 Supplementary Figures

1.1 Correlation between metacognitive sensitivity and postdecisional accumulation period

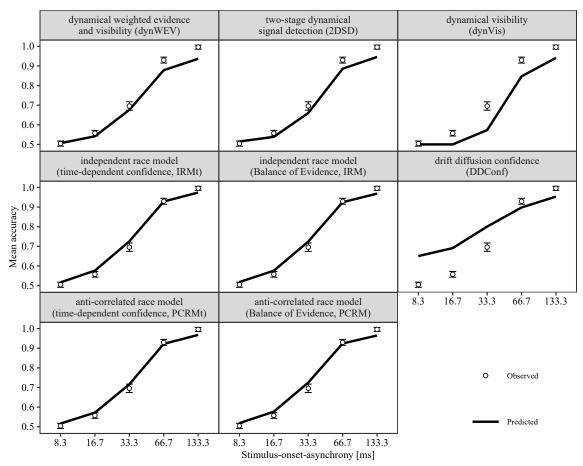
Supplementary Figure 1: Estimated postdecisional accumulation period in 2DSD and dynWEV against observed Type2-AUC



Note. Relationship of Type2-AUC as measure for metacognitive sensitivity and estimated postdecisional accumulation period τ for experiment 1 (top row) and experiment 2 (bottom row). The blue lines show the linear regression line for each panel. The correlation coefficients between Type2-AUC and fitted τ were .53 for 2DSD and .45 for dynWEV in experiment 1 and .49 and .38 for experiment 2 respectively.

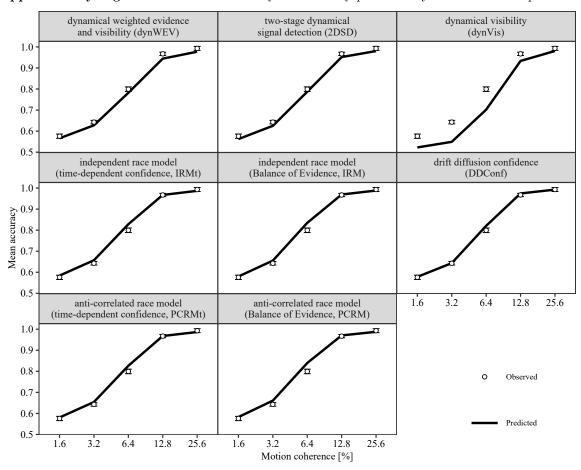
1.2 Accuracy across experimental conditions

Supplementary Figure 2: Observed accuracy vs. accuracy predicted by model fits for experiment 1



Note. Empirical (points) and fitted (lines) choice accuracy by stimulus-onset-asynchrony for different computational models (panels). Error bars represent within subject standard errors.

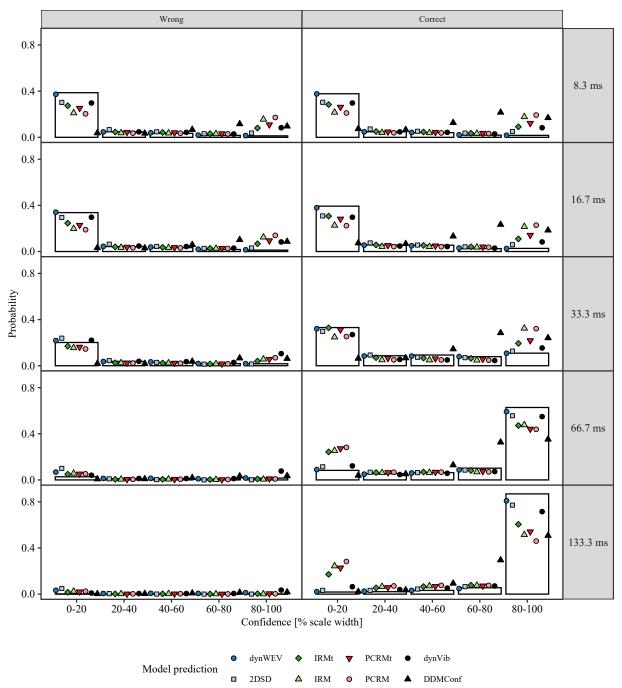
Supplementary Figure 3: Observed accuracy vs. accuracy predicted by model fits for experiment 2



Note. Empirical (points) and fitted (lines) choice accuracy by stimulus-onset-asynchrony for different computational models (panels). Error bars represent within subject standard errors.

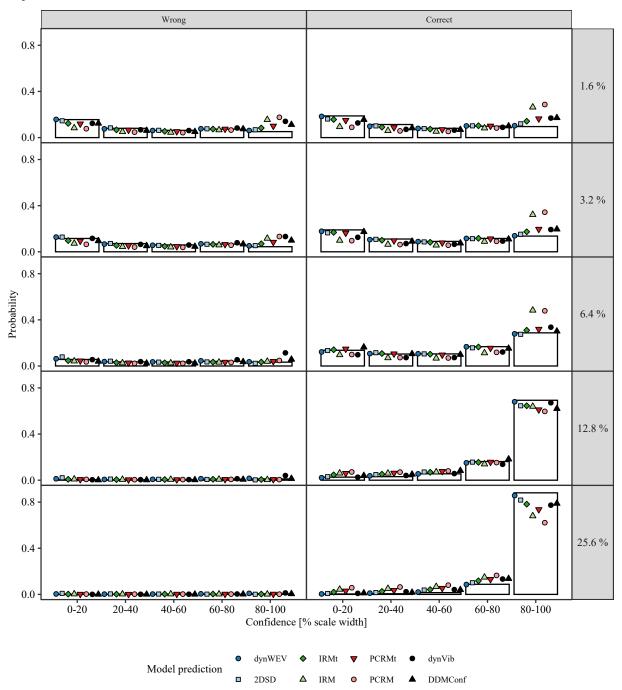
1.3 Distribution of accruacy and confidence responses

Supplementary Figure 4: Observed rating distribution vs. distribution predicted by model fits for experiment 1



Note. Empirical (bars) and fitted (points) distribution of choice and confidence responses across levels of stimulus discriminability (rows). Height of the bars represents the joint probability of a wrong/correct decision and a given confidence rating, i.e. bars in each row sum to 1.

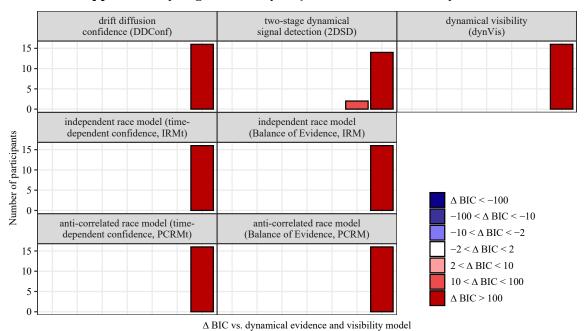
Supplementary Figure 5: Observed rating distribution vs. distribution predicted by model fits for experiment 2



Note. Empirical (bars) and fitted (points) distribution of choice and confidence responses across levels of stimulus discriminability (rows). Height of the bars represents the joint probability of a wrong/correct decision and a given confidence rating, i.e. bars in each row sum to 1.

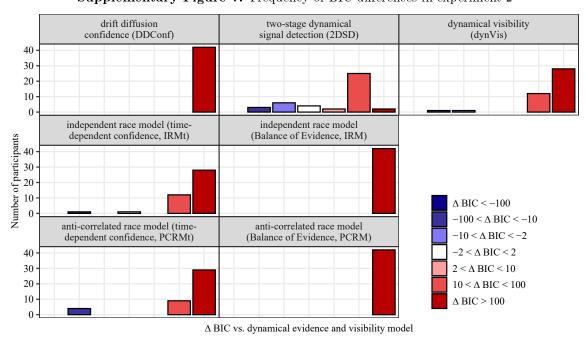
1.4 Frequency of magnitudes of BIC differences

Supplementary Figure 6: Frequency of BIC differences in experiment 1



Note. Plot shows the frequency of magnitudes in BIC differences across participants. For each participant the difference in BIC between model fits for dynWEV and the alternative models $(BIC_m - BIC_{dynWEV})$ for different models m is calculated and binned in magnitudes.

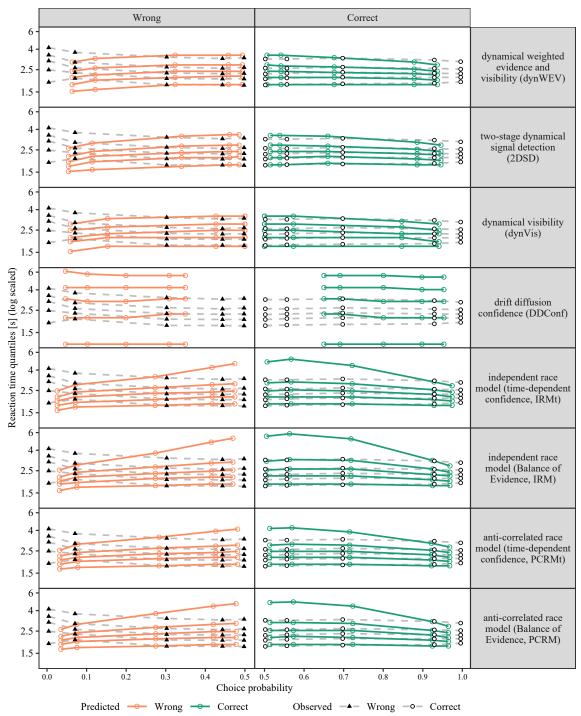
Supplementary Figure 7: Frequency of BIC differences in experiment 2



Note. Plot shows the frequency of magnitudes in BIC differences across participants. For each participant the difference in BIC between model fits for dynWEV and the alternative models $(BIC_m - BIC_{dynWEV})$ for different models m is calculated and binned in magnitudes.

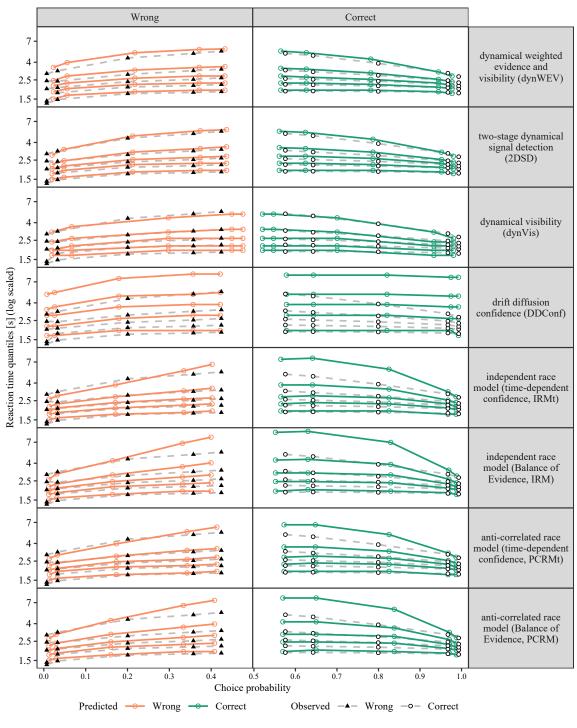
1.5 Response time quantiles across discriminability

Supplementary Figure 8: Quantile probability plots for experiment 1



Note. Response time quantiles against choice probability for different levels of stimulus-onset-asynchrony. The x-axis shows the choice probability as in standard quantile probability plots (Ratcliff & Smith, 2004) but split up in two columns, that means that in the left panels the x-axis show the probability of an error and in the right panels the probability of a correct response. This means that easier conditions are printed on the left and right borders and hard conditions are closer to the center as choice probabilities tend towards 50%. Colored lines and points show the model predictions. Dashed lines, triangles and black points show observed data. Quantiles are computed for pooled data.

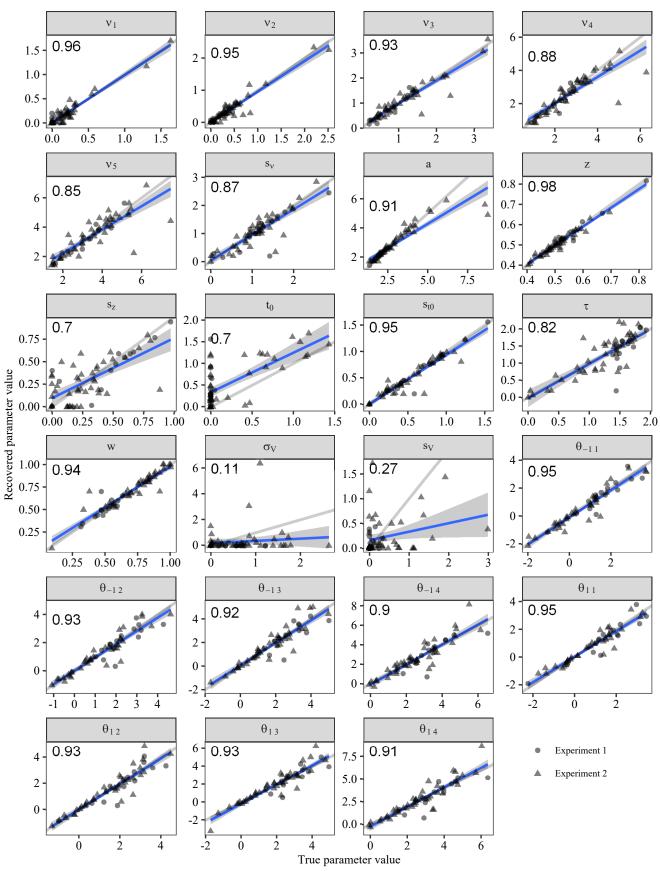
1.6 Parameter recovery for dynWEV



Supplementary Figure 9: Quantile probability plots for experiment 2

Note. Response time quantiles against choice probability for different levels of stimulus-onset-asynchrony. The x-axis shows the choice probability as in standard quantile probability plots (Ratcliff & Smith, 2004) but split up in two columns, that means that in the left panels the x-axis show the probability of an error and in the right panels the probability of a correct response. This means that easier conditions are printed on the left and right borders and hard conditions are closer to the center as choice probabilities tend towards 50%. Colored lines and points show the model predictions. Dashed lines, triangles and black points show observed data. Quantiles are computed for pooled data.

Supplementary Figure 10: Results for parameter recovery study for dynWEV



Note. True and recovered parameters from parameter recovery analysis. Gray lines represent the identity line. Blue lines with shaded area are linear regression lines with .95 confidence area. Numbers show Pearson correlation of true and fitted parameters. Parameter recovery was conducted by generating one artificial data set for each participant with previously fitted parameters and sample size as observed in the actual experiments. Afterwards, the model was fitted to these artificial data sets. Confidence criteria with an infinite value are not shown and were not considered in the computation of correlation. These values occur when the lowest or highest rating categories were not present in the data.

2 Supplementary Tables

2.1 Results for model comparison for all models

Supplementary Table 1: Results from quantitative mode comparison with dynWEV using BIC for experiment 1

Model	$M_{\Delta} (SD_{\Delta})$	BF_{10}	95% CI
2DSD	256(120)	5.2×10^{4}	[1.12, 2.93]
DDConf	5783(1717)	1.5×10^7	[1.94, 4.52]
dynVis	630(244)	$5.3 imes 10^5$	[1.43, 3.52]
IRM	1351(380)	3.0×10^7	[2.06, 4.76]
IRMt	892(245)	4.1×10^7	[2.11, 4.87]
PCRM	1362(383)	3.0×10^7	[2.06, 4.77]
PCRMt	1028(306)	1.4×10^7	[1.93, 4.50]

Note. M_{Δ} represents the mean BIC difference (with respective standard deviation in parentheses) to dynWEV and BF_{10} shows the Bayes factor for a Bayesian t-Test comparison with the associated posterior 95% equal-tailed credible interval on effect size in the last column. See main document for further details.

Supplementary Table 2: Results from quantitative mode comparison with dynWEV using BIC for experiment 2

Model	$M_{\Delta} \ (SD_{\Delta})$	BF_{10}	95% CI
2DSD	28(35)	3.3×10^{3}	[0.44, 1.13]
DDConf	1796(1059)	1.1×10^{11}	[1.18, 2.14]
dynVis	269(230)	5.2×10^6	[0.75, 1.54]
IRM	605(409)	2.1×10^{9}	[1.01, 1.89]
IRMt	170(135)	3.2×10^7	[0.82, 1.64]
PCRM	550(400)	2.9×10^8	[0.92, 1.77]
PCRMt	191(179)	6.2×10^5	[0.66, 1.42]

Note. M_{Δ} represents the mean BIC difference (with respective standard deviation in parentheses) to dynWEV and BF_{10} shows the Bayes factor for a Bayesian t-Test comparison with the associated posterior 95% equal-tailed credible interval on effect size in the last column. See main document for further details.

2.2 Summary tables for parameter fits

Supplementary Table 3: Mean and standard deviation of parameter fits for experiment 1

				Model	del			
Parameter	dynWEV	2DSD	DDConf	dynVis	IRM	IRMt	PCRM	PCRMt
$ u_1$	0.03(0.05)	0.05(0.08)	0.52(0.05)	0.00(0.00)	0.03(0.04)	0.03(0.04)	0.04(0.05)	0.03(0.04)
ν_2	0.17(0.16)	0.11(0.09)	0.68(0.09)	0.00(0.00)	0.12(0.09)	0.13(0.10)	0.14(0.11)	0.15(0.12)
ν_3	0.78(0.42)	0.49(0.33)	1.22(0.35)	0.24(0.23)	0.42(0.25)	0.48(0.30)	0.49(0.29)	0.54 (0.33)
V_4	2.18(0.82)	1.76(0.72)	2.12(0.79)	1.40(0.64)	$1.54\ (0.60)$	1.73(0.60)	1.59(0.60)	1.81(0.74)
$ u_5 $	3.00(1.06)	2.51 (0.73)	3.39(1.41)	2.34(0.95)	2.28(0.68)	2.60(0.64)	2.18(0.63)	2.57(0.78)
SV	1.21(0.63)	0.30(0.29)	0.19(0.36)	0.55(0.38)	I	I	I	I
ಹ	1.96(0.31)	1.80(0.24)	1.28(0.16)	1.81 (0.29)	ı	I	I	I
Z	0.55(0.09)	0.55(0.09)	0.50(0.08)	0.54(0.08)	I	ı	ı	I
S_{Z}		0.42(0.34)	0.14(0.16)	0.05(0.19)	I	I	I	I
A	I	I	I	l	0.93(0.23)	0.80(0.21)	1.01(0.21)	0.87(0.21)
В	I	I	I	I	0.79(0.23)	0.69(0.19)	0.87(0.23)	0.75(0.18)
$\theta_{1,1}$		2.45(1.34)	0.75(0.16)	0.06(0.27)	1.03(0.40)	1.68(0.62)	1.41(0.41)	1.75(0.52)
$\theta_{1,2}$	2.13(1.24)	3.07(1.17)	0.54(0.13)	0.18(0.21)	1.20(0.45)	1.93(0.65)	1.56(0.42)	1.96(0.53)
$\theta_{1,3}$		3.70(0.98)	0.34(0.09)	0.32(0.25)	1.38(0.56)	2.15(0.68)	1.71 (0.46)	2.14(0.53)
$\theta_{1,4}$		4.14(0.79)	0.14(0.05)	0.36(0.13)	1.39(0.29)	2.26(0.52)	1.74 (0.28)	2.23(0.43)
$\theta_{-1,1}$		-2.10(0.66)	0.80(0.16)	0.00(0.40)	1.13(0.41)	1.81(0.53)	1.50(0.39)	1.85(0.46)
$\theta_{-1,2}$		-1.78(0.94)	0.58(0.13)	0.15(0.33)	1.32(0.38)	2.09(0.50)	1.67(0.34)	2.09(0.42)
$\theta_{-1,3}$	2.76(0.97)	-1.25(0.95)	0.36(0.10)	0.30(0.35)	1.52(0.39)	2.40(0.45)	1.85(0.33)	2.32(0.39)
$\theta_{-1,4}$		-0.71(1.00)	0.16(0.06)	0.32(0.09)	1.60(0.27)	2.58(0.39)	1.92(0.25)	2.46(0.37)
t_0		0.02(0.07)	0.41 (0.26)	1.50(0.16)	1.51 (0.16)	1.54(0.17)	1.51(0.17)	1.52(0.18)
s_{t0}	0.64(0.38)	0.62(0.37)	5.12(1.39)	0.74(0.40)	0.71(0.33)	0.78(0.38)	0.60(0.31)	0.76(0.33)
T	1.57(0.17)	1.56(0.19)	l	I	I	I	I	I
W	0.50(0.09)	I	ı	ı	I	I	I	I
As	0.06(0.10)	l	l	I	ı	I	I	1
Δho	0.53(0.36)	I	I	l	I	ı	I	1
w_x	I	I	I	I	I	0.19(0.24)	I	0.46(0.28)
w_{rt}	1	1	1	1	1	0.10(0.14)	1	0.03(0.07)
w_{int}	-	-	I	1	1	0.78(0.27)	1	0.52 (0.29)

Supplementary Table 4: Mean and standard deviation of parameter fits for experiment 2

				Model	lel			
Parameter	dynWEV	2DSD	${ m DDConf}$	dynVis	$_{ m IRM}$	IRMt	PCRM	PCRMt
$ u_1 $	0.24(0.31)	0.19(0.29)	0.33(0.38)	0.05(0.15)	0.09(0.10)	0.12(0.14)	0.12(0.13)	0.13(0.15)
ν_2	0.49(0.49)	0.39(0.47)	0.62(0.53)	0.12(0.27)	0.19(0.17)	0.22(0.21)	0.23(0.21)	0.25(0.23)
ν_3	_	0.99(0.71)	1.70(0.88)	0.53(0.44)	0.55(0.29)	0.62(0.31)	0.66(0.33)	0.70(0.35)
$ u_4$	_	2.43(1.18)	4.88(1.73)	1.83(0.75)	1.77(0.61)	1.82(0.56)	1.86(0.60)	1.95(0.61)
$ u_5 $	_	3.27(1.30)		2.80(0.97)	2.58(0.81)	2.64(0.83)	2.58(0.77)	2.78(0.94)
$s\nu$	_	0.62(0.44)	0.71(0.53)	0.50(0.40)	·	·		
ಣ	3.23(1.64)	2.99(1.52)	1.47(0.71)	2.72(1.00)	I	I	I	I
Z	_	0.51 (0.06)	0.50(0.10)	0.51(0.05)	I	I	1	ı
s_z	0.28(0.23)	0.27(0.22)	0.03(0.11)	0.04(0.07)	I	I	I	I
A	I	1	I	1	1.29(0.45)	1.08(0.36)	1.37(0.45)	1.17(0.38)
В	I	1	I	I	1.29(0.54)	1.08(0.42)	1.36(0.54)	1.17(0.44)
$ heta_{1,1}$	0.36(1.55)	3.03(2.34)	0.59(0.39)	-1.35 (1.68)	0.92(0.75)	1.12(0.65)	1.38(0.89)	1.36(0.76)
$\theta_{1,2}$	$1.14\ (1.56)$	3.73(2.40)	0.38(0.25)	-0.57 (1.19)	1.26(0.80)	1.52(0.73)	1.74(0.91)	1.77(0.78)
$\theta_{1,3}$	1.70(1.55)	4.26(2.35)	0.29(0.21)	-0.10(0.91)	1.52(0.80)	1.86(0.74)	2.01(0.86)	2.12(0.74)
$\theta_{1,4}$	(1	5.12(2.31)	0.20(0.17)	0.54(0.90)	1.95(0.76)	2.42(0.77)	2.40(0.82)	2.62(0.71)
$\theta_{-1,1}$	0.21(1.54)	-2.05(1.34)	0.62(0.34)	-1.75(2.31)	0.99(0.71)	1.22(0.63)	1.47 (0.87)	
$\theta_{-1,2}$	1.02(1.44)	-1.12(1.30)	0.40(0.22)	-0.81 (1.46)		1.63(0.72)	1.81(0.89)	1.88(0.74)
$\theta_{-1,3}$	1.56(1.48)	-0.56(1.31)	0.30(0.20)	-0.35(1.19)	1.57(0.74)	1.98(0.72)	2.07(0.81)	2.21(0.67)
$\theta_{-1,4}$	2.52(1.50)	0.20(1.28)	0.19(0.14)	0.44(0.76)	1.96(0.72)	2.49(0.74)	2.41(0.79)	2.67(0.63)
t_0	0.32(0.44)	0.23(0.39)	1.31 (0.30)	1.43(0.21)	1.43(0.22)	1.48(0.21)	1.44(0.23)	1.45(0.21)
s_{t0}	_	0.53(0.35)	6.52(4.17)	0.55(0.38)	0.55(0.46)	0.63(0.39)	0.47 (0.39)	0.63(0.36)
٢	1.17(0.51)	1.26(0.50)						·
W	0.74(0.19)	1	I	I	I	I	I	I
SV	0.48(0.63)	I	I	1	I	I	I	I
Δho	0.68(0.67)	I	I	1	I	I	1	l
w_x	ļ	1	I	1	1	0.11(0.14)	I	0.26(0.22)
w_{rt}	I	1	I	1	1	$0.34\ (0.32)$	1	0.15(0.28)
w_{int}			_	_		0.75(0.24)	-	0.68(0.23)

3 Mathematical formulae for likelihood functions

This section includes the derivation of the likelihood functions of all models used for model fitting together with references for cited formulas.

3.1 Notational notes

Some frequently used mathematical symbols and notational conventions are defined here. Beside these, we will used \mathcal{C} for terms that are not of specific interest and do not change in longer equations and note parameter dependencies only if they are explicitly relevant.

 \mathbb{R} set of real numbers

 $X \sim \mathcal{N}(\mu, \sigma^2)$ the random variable X follows a normal

distribution with mean μ and standard deviation σ

 $\varphi(x|\mu,\sigma^2)$, $\Phi(x|\mu,\sigma^2)$ the pdf and cdf of a random variable with

mean μ and standard deviation σ

3.2 Useful identities

Before reporting the computations in the models, it may serve practical to state two frequently used identities concerning products and integrals involving the Gaussian pdf and cdf. First, the product of two normal density functions with the mean equal to the value of the other variable may be rewritten as

$$\varphi(x|y,\sigma_x^2)\varphi(y|\mu_0,\sigma_0^2) = \varphi\left(y \left| \frac{\sigma_0^2 x + \sigma_x^2 \mu_0}{\sigma_0^2 + \sigma_x^2}, \frac{\sigma_0^2 \sigma_x^2}{\sigma_0^2 + \sigma_x^2} \right) \varphi(x|\mu_0,\sigma_0^2 + \sigma_x^2).$$
 (1)

Similarly, because $\varphi(x|\mu,\sigma^2)=\varphi(\mu|x,\sigma^2)$, we can also write

$$\varphi(x|\mu,\sigma^2)\varphi(x|m,s^2) = \varphi\left(x\left|\frac{\sigma^2m + s^2\mu}{\sigma^2 + s^2}, \frac{\sigma^2s^2}{\sigma^2 + s^2}\right)\varphi(\mu|m,s^2 + \sigma^2)\right)$$
(2)

and just slightly different

$$\varphi(x|\mu,\sigma^2)\varphi(\alpha x|m,s^2) = \varphi(x|\mu,\sigma^2)\varphi(m|\alpha x,s^2)$$
(3)

$$= \varphi\left(x \middle| \frac{\sigma^2 \alpha m + s^2 \mu}{\sigma^2 \alpha^2 + s^2}, \frac{\sigma^2 s^2}{\sigma^2 \alpha^2 + s^2}\right) \varphi(m | \alpha \mu, s^2 + \sigma^2 \alpha^2). \tag{4}$$

3.3 Response time distribution in the simple drift diffusion model

In the next step, we report the density of reaction times and decisions in the simple drift diffusion model without inter-trial variability of parameters. The density can be found in the literature (Voss et al., 2004; Wabersich & Vandekerckhove, 2014). Here, we use the density as given in Wabersich and Vandekerckhove (2014), which is close to the computational implementation in the analyses. For *lower* responses (R=-1), with parameters δ (drift rate), a (boundary separation), and z (rel. starting point) it is

$$f_H(R = -1, t | \delta, a, z) = \frac{1}{a^2} \exp\left[-az\delta - \frac{1}{2}\delta^2 t\right] f\left(\frac{t}{a^2} | z\right), \tag{5}$$

with f depending on the size of t. Note, that the diffusion constant is set to 1. For *upper* responses, making use of the symmetry we get $f_H(t, R = 1 | \delta, a, z) = f_H(t, R = -1 | -\delta, a, 1 - z)$. The function f depends on the approximation used for the density, which is more or less efficient for different regions of decision times. For large t, we use

$$f_L(u|z) = \pi \sum_{k=1}^{\infty} k \exp\left[-\frac{k^2 \pi^2 u}{2}\right] \sin(k\pi z)$$

and for small t the approximation

$$f_S(u|z) = \frac{1}{\sqrt{2\pi u^3}} \sum_{k=-\infty}^{\infty} (2k+z) \exp\left[-\frac{(2k+z)^2}{2u}\right]$$

is more efficient. Note, that the drift rate occurs only in the first exponential of the density. For some formulae it is useful to write the terms including the drift rate δ with a normal density, i.e.

$$f_H(R = -1, t | \delta, a, z) = \frac{1}{a^2} \exp\left[-\frac{1}{2}(t\delta^2 + 2\delta az)\right] f\left(\frac{t}{a^2} | z\right)$$

$$= \frac{\sqrt{2\pi}}{a^2 \sqrt{t}} \exp\left[\frac{a^2 z^2}{2t}\right] \varphi\left(\delta \left| -\frac{az}{t}, \frac{1}{t} \right| f\left(\frac{t}{a^2} | z\right)\right). \tag{6}$$

3.4 Response distribution in 2DSD

The response consists of multiple components. Specifically, we observe the response time, t, the decision outcome R and the discrete confidence rating, denoted as C. The confidence measure is defined as $c := X(T_{dec} + \tau)$ in accordance with Pleskac and Busemeyer (2010). Let θ_R be the respective decision threshold, i.e. $\theta_R = a$, if R = 1 and $\theta_R = 0$, if R = -1. Then $c|R, \delta \sim \mathcal{N}(\theta_R + \delta \tau, \tau)$. The distribution of a response R = -1, given a fixed drift rate δ , is

$$p(R = -1, t, c|\delta) = f_H(R = -1, t|\delta) \ p(c|\delta, R = -1)$$
$$= f_H(R = -1, t|\delta) \ p(c|\delta, R = -1)$$
$$= f_H(R = -1, t|\delta) \ \varphi(c|\tau\delta, \tau).$$

The distribution for the opposite decision is derived from this by symmetry and is given below. For a variable drift rate $\delta \sim \mathcal{N}(\nu, s_{\nu}^2)$, we may compute the inner integral as

$$\begin{split} p(R = -1, t, c) &= \int_{\mathbb{R}} p(R = -1, t, c | \delta) \varphi(\delta | \nu, s_{\nu}^{2}) d\delta \\ &= \int_{\mathbb{R}} f_{H}(R = -1, t | \delta) \varphi(c | \tau \delta, \tau) \varphi(\delta | \nu, s_{\nu}^{2}) d\delta \\ &= \int_{\mathbb{R}} \frac{\sqrt{2\pi}}{a^{2} \sqrt{t}} \exp\left[\frac{a^{2} z^{2}}{2t}\right] \varphi\left(\delta \left| -\frac{az}{t}, \frac{1}{t}\right) f\left(\frac{t}{a^{2}} \middle| z\right) \\ &\times \varphi(\tau \delta | c, \tau) \varphi(\delta | \nu, s_{\nu}^{2}) d\delta \\ &= \frac{\sqrt{2\pi}}{a^{2} \sqrt{t}} \exp\left[\frac{a^{2} z^{2}}{2t}\right] f\left(\frac{t}{a^{2}} \middle| z\right) \\ &\times \int_{\mathbb{R}} \varphi(\tau \delta | c, \tau) \varphi\left(\delta \left| \frac{-az s_{\nu}^{2} + \nu}{1 + s_{\nu}^{2} t}, \frac{s_{\nu}^{2}}{1 + s_{\nu}^{2} t}\right) \varphi\left(\nu \left| \frac{-az}{t}, \frac{1/t + s_{\nu}^{2}}{t}\right) d\delta \\ &= \frac{\sqrt{2\pi}}{a^{2} \sqrt{t}} \exp\left[\frac{a^{2} z^{2}}{2t}\right] f\left(\frac{t}{a^{2}} \middle| z\right) \frac{1}{\sqrt{2\pi(1/t + s_{\nu}^{2})}} \exp\left[-\frac{(\nu + \frac{az}{t})^{2}}{2(1/t + s_{\nu}^{2})}\right] \\ &\times \int_{\mathbb{R}} \varphi\left(\delta \left| \frac{c\tau \frac{s_{\nu}^{2}}{1 + s_{\nu}^{2} t} + \tau}{\tau^{2} \frac{s_{\nu}^{2}}{1 + s_{\nu}^{2} t} + \tau}, \frac{\tau \frac{s_{\nu}^{2}}{1 + s_{\nu}^{2} t}}{\tau^{2} \frac{s_{\nu}^{2}}{1 + s_{\nu}^{2} t} + \tau}\right) \varphi\left(c \left| \tau \frac{-az s_{\nu}^{2} + \nu}{1 + s_{\nu}^{2} t}, \tau^{2} \frac{s_{\nu}^{2}}{1 + s_{\nu}^{2} t} + \tau\right) d\delta \\ &= \frac{1}{a^{2} \sqrt{1 + s_{\nu}^{2} t}} f\left(\frac{t}{a^{2}} \middle| z\right) \exp\left[\frac{a^{2} z^{2} (1/t + s_{\nu}^{2}) - t(\nu + \frac{az}{t})^{2}}{2(1 + s_{\nu}^{2} t)}\right] \\ &\times \varphi\left(c \left| \tau \frac{-az s_{\nu}^{2} + \nu}{1 + s_{\nu}^{2} t}, \tau^{2} \frac{s_{\nu}^{2}}{1 + s_{\nu}^{2} t} + \tau\right) \\ &= \frac{1}{a^{2} \sqrt{1 + s_{\nu}^{2} t}} f\left(\frac{t}{a^{2}} \middle| z\right) \exp\left[\frac{a^{2} z^{2} s_{\nu}^{2} - t\nu^{2} - 2\nu az}{2(1 + s_{\nu}^{2} t)}\right] \end{aligned}$$

$$(8)$$

$$\times \varphi\left(c|\mu_c,\sigma_c^2\right)$$

with $\mu_c = \frac{\tau(\nu - azs_{\nu}^2)}{s_{\nu}^2 t + 1}$ and $\sigma_c^2 = \frac{\tau(s_{\nu}^2 t + s_{\nu}^2 \tau + 1)}{(s_{\nu}^2 t + 1)}$. For categorical confidence judgments, we have C = i, only if, $c \in [\vartheta_{R,i}, \vartheta_{R,i+1}]$, so integrating the probability density above over this interval results in

$$p(R = -1, t, C = i) = \frac{1}{a^2 \sqrt{s_{\nu}^2 t + 1}} f\left(\frac{t}{a^2} \middle| z\right) \exp\left[\frac{a^2 z^2 s_{\nu}^2 - t\nu^2 - 2\nu az}{2(1 + s_{\nu}^2 t)}\right] \times \left(\Phi(\vartheta_{-1, i+1} \middle| \mu_c, \sigma_c^2) - \Phi(\vartheta_{-1, i} \middle| \mu_c, \sigma_c^2)\right).$$
(9)

For completeness, we also give the formula for the other choice, R=1, which is

$$\begin{split} p(R=1,t,C=i) &= \int_{\vartheta_{1,i}}^{\vartheta_{1,i+1}} \int_{\mathbb{R}} p(R=1,t|\delta,a,z) p(c|R=1,\delta) p(\delta|\nu,s_{\nu}) d\delta dc \\ &= \int_{\vartheta_{1,i}}^{\vartheta_{1,i+1}} \int_{\mathbb{R}} p(t,R=-1|-\delta,a,1-z) p(c|R=1,\delta) p(-\delta|-\nu,s_{\nu}) d\delta dc \\ &= \sum_{\tilde{c}:=a-c} \int_{a-\vartheta_{1,i+1}}^{a-\vartheta_{1,i}} \int_{\mathbb{R}} p(t,R=-1|-\delta,a,1-z) p(\tilde{c}|R=-1,-\delta) p(-\delta|-\nu,s_{\nu}) d\delta dc \\ &= \frac{1}{a^2 \sqrt{1+s_{\nu}^2 t}} f\left(\frac{t}{a^2} \bigg| z\right) \exp\left[\frac{a^2 (1-z)^2 s_{\nu}^2 - t \nu^2 - 2(-\nu) a(1-z)}{2(1+s_{\nu}^2 t)}\right] \\ &\qquad \times \left(\Phi(a-\vartheta_{-1,i}|\mu_{\tilde{c}},\sigma_c^2) - \Phi(a-\vartheta_{-1,i+1}|\mu_{\tilde{c}},\sigma_c^2)\right), \end{split}$$

where $\mu_{\tilde{c}} = \frac{\tau(-v-a(1-z)s_{\nu}^2)}{s_{\nu}^2t+1}$. Note, that $\theta_1 = a$. We therefore may use the symmetry $p(t,R=1,C=i|\theta_{1,i},\vartheta_{1,i+1},\nu,a,z) = p(t,R=-1,C=i|a-\vartheta_{1,i+1},a-\vartheta_{1,i},-\nu,a,1-z)$, and use eqn. (8) for computations. At this point, it is obvious that for non-symmetric confidence thresholds the parameters $\vartheta_{1,i}$ could be shifted by $\theta_1 = a$ to drop this addend in the computations.

3.5 Response distribution in dynWEV

For simplicity and to facilitate the notion of symmetric confidence thresholds in the model, we will use $e_c = R(X(T_{dec} + \tau) - \theta_R)$ instead of $X(T_{dec} + \tau)$ in the computation of confidence, here. Thus, the confidence measure in dynWEV is given by $c = we_c + (1 - w)V(T_{dec} + \tau)$. The two formulations are equivalent for non-symmetric thresholds, but it is more intuitive to use the e_c , here, as it represents the amount of evidence in favor of the previously made decision accumulated in the post-decisional period. Positive e_c thus implies higher confidence than a negative e_c . Note, that e_c and $V(T_{dec} + \tau)$ are conditionally independent given T_{dec} and both follow a normal distribution. The decision time itself depends on δ . More precisely, $V(t+\tau) \sim \mathcal{N}(|\nu|(t+\tau), \sigma_V^2(t+\tau)^2 + s_V^2(t+\tau))$ and $e_c|R, t, \delta \sim \mathcal{N}(R\delta\tau, \tau)$. We can conclude with slight abuse of notation

$$\begin{split} c|R,t,\delta &\sim w \mathcal{N} \left(R\delta \tau,\tau\right) \; + \; (1-w) \mathcal{N} \left(|\nu|(t+\tau),\sigma_V^2(t+\tau)^2 + s_V^2(t+\tau)\right) \\ &= w R \tau \mathcal{N} \left(\delta + \frac{(1-w)(t+\tau)}{Rw\tau} |\nu|,\; \frac{1}{\tau} + \frac{(1-w)^2(\sigma_V^2(t+\tau)^2 + s_V^2(t+\tau))}{w^2\tau^2}\right) \\ &= w R \tau \mathcal{N} \left(\delta + \frac{(1-w)(t+\tau)}{Rw\tau} |\nu|,\; \frac{1}{\tau} + \frac{(1-w)^2(\sigma_V^2(t+\tau)^2 + s_V^2(t+\tau))}{w^2\tau^2}\right). \end{split}$$

Let $q = wR\tau$, $E = \frac{(1-w)(t+\tau)}{Rw\tau}|\nu|$, and $S^2 = \frac{1}{\tau} + \frac{(1-w)^2(\sigma_V^2(t+\tau)^2 + s_V^2(t+\tau)}{w^2\tau^2}$. With similar computations as for 2DSD, we get

$$P(R = -1, t, c) = \int_{\mathbb{R}} f_H(R = -1, t|\delta) p(c|R, t, \delta) \varphi(\delta|\nu, s_\nu^2) d\delta$$

$$= \int_{\mathbb{R}} \frac{\sqrt{2\pi}}{a^2 \sqrt{t}} \exp\left[\frac{a^2 z^2}{2t}\right] \varphi\left(\delta \middle| -\frac{az}{t}, \frac{1}{t}\right) f\left(\frac{t}{a^2}\middle| z\right)$$

$$\times \frac{1}{w\tau} \varphi(\delta | c/q - E, S^2) \varphi(\delta | \nu, s_{\nu}^2) d\delta$$

$$= \frac{1}{a^2 \sqrt{1 + s_{\nu}^2 t}} f\left(\frac{t}{a^2}\middle| z\right) \exp\left[\frac{a^2 z^2 s_{\nu}^2 - t\nu^2 - 2\nu az}{2(1 + s_{\nu}^2 t)}\right]$$

$$\times \frac{1}{w\tau} \int_{\mathbb{R}} \varphi(\delta | c/q - E, S^2) \varphi\left(\delta \middle| \frac{-az s_{\nu}^2 + \nu}{1 + s_{\nu}^2 t}, \frac{s_{\nu}^2}{1 + s_{\nu}^2 t}\right) d\delta$$

$$= \mathcal{C} \int_{\mathbb{R}} \varphi\left(\delta \middle| \frac{(c/q - E) \frac{s_{\nu}^2}{1 + s_{\nu}^2 t} + \frac{-az s_{\nu}^2 + \nu}{1 + s_{\nu}^2 t} S^2}{S^2 + \frac{s_{\nu}^2}{1 + s_{\nu}^2 t}}, \frac{S^2 \frac{s_{\nu}^2}{1 + s_{\nu}^2 t}}{S^2 + \frac{s_{\nu}^2}{1 + s_{\nu}^2 t}}\right) d\delta$$

$$\times \frac{1}{w\tau} \varphi\left(c/q - E \middle| \frac{-az s_{\nu}^2 + \nu}{1 + s_{\nu}^2 t}, S^2 + \frac{s_{\nu}^2}{1 + s_{\nu}^2 t}\right)$$

$$= \mathcal{C} \varphi\left(c \middle| \mu_c, \sigma_c^2\right) \tag{10}$$

with $\mu_c = \frac{wR\tau(-azs_{\nu}^2+\nu)}{1+s_{\nu}^2t} + (1-w)(t+\tau)|\nu|$ and $\sigma_c^2 = (1-w)^2(\sigma_V^2(t+\tau)^2 + s_V^2(t+\tau)) + w^2\tau + \frac{s_{\nu}^2w^2\tau^2}{1+s_{\nu}^2t}$. For a discrete confidence judgment, the density is again integrated with respect to c. Furthermore, the symmetry property $p(R=1,t,c|\vartheta_{1,i},\vartheta_{1,i+1},\nu,a,z) = p(R=1,t,c|\vartheta_{1,i},\vartheta_{1,i+1},-\nu,a,1-z)$ holds.

3.6 Response distribution in race models

Here, we reproduce previously published closed forms for the joint probability density of process k winning, the decision time t and the state of the losing accumulator x_j , denoted as $g_k(t, x_j)$ (Moreno-Bote, 2010). We cite only the formulae in this work and refer to the original article for how to achieve these solutions. For better readability we assume without loss of generality, that k = 1 and j = 2 and omit the indices where they are not necessary. It is then

$$g(t, x_2) = -\frac{\sigma^2}{2} \frac{\partial}{\partial x_1} P(x_1, x_2, t)|_{x_1 = 0},$$

where $P(x_1, x_2, t) = P_e(x_1, x_2, t | x_0 = (A, B)) + \sum_{i=1}^{\ell} c_i P_e(x_1, x_2, t | x_0 = x_i')$ is the solution to the Focker-Planck-Equation with the corresponding boundary conditions attained with the methods of images. For the two choices of ρ , considered one needs a different number of images, such that $\ell = 3$, for $\rho = 0$ and $\ell = 5$ for $\rho = -0.5$. The probability $P_e(x_1, x_2, t | x_0 = x')$ is the probability distribution of unconstrained process states given time t and starting point x_i and is given by a bivariate Gaussian. It is expressed in terms of the transformation $u = x_1 + x_2$, $v = x_1 - x_2$, namely

$$P_e(x_1, x_2, t | x_0 = x') = \frac{1}{\pi \sigma_u \sigma_v t} \exp \left[-\frac{((x_1 + x_2) - (x_1' + x_2') - \mu_u t)^2}{2\sigma_u^2 t} - \frac{((x_1 - x_2) - (x_1' - x_2') - \mu_v t)^2}{2\sigma_v^2 t} \right].$$

Let in this situation $\tilde{x}_1 = -x_1' - \mu_1 t$ and $\tilde{x}_2 = x_2 - x_2' - \mu_2 t$. Taking the derivative with respect to x_1 and then setting $x_1 = 0$ gives

$$\frac{\partial}{\partial x_1} P_e(x_1, x_2, t | x_0 = x'))|_{x_1 = 0} = -\frac{1}{\pi \sigma_u \sigma_v t^2} \left(\frac{\tilde{x}_2 + \tilde{x}_1}{\sigma_u^2} - \frac{\tilde{x}_2 - \tilde{x}_1}{\sigma_v^2} \right) \exp\left[-\frac{(\tilde{x}_2 + \tilde{x}_1)^2}{2\sigma_u^2 t} - \frac{(\tilde{x}_2 - \tilde{x}_1)^2}{2\sigma_v^2 t} \right]. \tag{11}$$

For the two choices of ρ the image starting points x'_i and coefficients c_i are given in Moreno-Bote (2010). We report them in Table 5. It is important to note here, that c_i is independent of both state components x_1 and x_2 for all i. Using $x'_0 = (A, B)$ and $c_0 = 1$, we may write

$$g_1(t, x_2) = -\frac{\sigma^2}{2} \sum_{i=0}^{\ell} c_i \frac{\partial}{\partial x_1} P_e(x_1, x_2, t | x_0 = x_i'))|_{x_1 = 0}.$$

The variance is $\sigma_u^2 = \sigma_v^2 = 2\sigma^2$, if $\rho = 0$ and $\sigma_u^2 = \sigma^2$, $\sigma_v^2 = 3\sigma^2$, if $\rho = -0.5$. Thus, we split the further computations for the two cases.

Supplementary Table 5: Image Points and Coefficients in RM distribution functions for the Race Models of Confidence.

3.7 Case: $\rho = 0$ (IRM and IRMt)

This case is a bit shorter, because in (11) the \tilde{x}_2 in the factor cancels such that the state-time distribution gets

$$g_1(t, x_2) = -\frac{\sigma^2}{2} \sum_{i=0}^3 c_i \frac{-1}{2\pi\sigma^2 t^2} \frac{2\tilde{x}_{1,i}}{2\sigma^2} \exp\left[-\frac{2(\tilde{x}_{1,i}^2 + \tilde{x}_{2,i}^2)}{4\sigma^2 t}\right]$$
$$= \frac{1}{4\pi\sigma^2 t^2} \sum_{i=0}^3 c_i \tilde{x}_{1,i} \exp\left[-\frac{\tilde{x}_{1,i}^2}{2\sigma^2 t}\right] \exp\left[-\frac{\tilde{x}_{2,i}^2}{2\sigma^2 t}\right],$$

which we may integrate in x_2 to get discrete confidence judgments according to the balance of evidence hypothesis:

$$\begin{split} P(t,x_2 \in [\vartheta_1,\vartheta_2]) &= \int_{\vartheta_1}^{\vartheta_2} g_1(t,x_2) dx_2 \\ &= \frac{1}{4\pi\sigma^2 t^2} \sum_{i=0}^3 c_i \tilde{x}_{1,i} \exp\left[-\frac{\tilde{x}_{1,i}^2}{2\sigma^2 t}\right] \int_{\vartheta_1}^{\vartheta_2} \exp\left[-\frac{(x_2 - x_{i,2} - \mu_2 t)^2}{2\sigma^2 t}\right] dx_2 \\ &= \frac{1}{4\pi\sigma^2 t^2} \sum_{i=0}^3 c_i \tilde{x}_{1,i} \exp\left[-\frac{\tilde{x}_{1,i}^2}{2\sigma^2 t}\right] \sqrt{\frac{\pi\sigma^2 t}{2}} \\ &\qquad \times \left[\operatorname{erf}\left(\frac{\vartheta_2 - x_{i,2} - \mu_2 t}{\sqrt{2\sigma^2 t}}\right) - \operatorname{erf}\left(\frac{\vartheta_1 - x_{i,2} - \mu_2 t}{\sqrt{2\sigma^2 t}}\right) \right] \\ &= \frac{1}{4\sqrt{2\pi\sigma^2 t^3}} \sum_{i=0}^3 c_i \tilde{x}_{1,i} \exp\left[-\frac{\tilde{x}_{1,i}^2}{2\sigma^2 t}\right] \left[\operatorname{erf}\left(\frac{\vartheta_2 - x_{i,2} - \mu_2 t}{\sqrt{2\sigma^2 t}}\right) - \operatorname{erf}\left(\frac{\vartheta_1 - x_{i,2} - \mu_2 t}{\sqrt{2\sigma^2 t}}\right) \right]. \end{split}$$

Here, we see that σ acts as a scaling factor, which we set to 1 without loss of generality. Now, substituting the values from Table 5 and setting $\sigma = 1$, we get

$$P(t,x_{2} \in [\vartheta_{1},\vartheta_{2}])(4\sqrt{2\pi t^{3}}) =$$

$$= (-A - \mu_{1}t) \exp\left[-\frac{(A + \mu_{1}t)^{2}}{2t}\right] \left\{ \left[\operatorname{erf}\left(\frac{\vartheta_{2} - B - \mu_{2}t}{\sqrt{2t}}\right) - \operatorname{erf}\left(\frac{\vartheta_{1} - B - \mu_{2}t}{\sqrt{2t}}\right)\right] - \exp\left[-2B\mu_{2}\right] \left[\operatorname{erf}\left(\frac{\vartheta_{2} + B - \mu_{2}t}{\sqrt{2t}}\right) - \operatorname{erf}\left(\frac{\vartheta_{1} + B - \mu_{2}t}{\sqrt{2t}}\right)\right] \right\}$$

$$- \exp\left[-2A\mu_{1}\right] (A - \mu_{1}t) \exp\left[-\frac{(-A + \mu_{1}t)^{2}}{2t}\right] \left\{ \ldots \text{same as above...} \right\}$$

$$= (-A - \mu_{1}t - A + \mu_{1}t) \exp\left[-\frac{(A + \mu_{1}t)^{2}}{2t}\right] \left\{ \left[\operatorname{erf}\left(\frac{\vartheta_{2} - B - \mu_{2}t}{\sqrt{2t}}\right) - \operatorname{erf}\left(\frac{\vartheta_{1} - B - \mu_{2}t}{\sqrt{2t}}\right)\right] - \exp\left[-2B\mu_{2}\right] \left[\operatorname{erf}\left(\frac{\vartheta_{2} + B - \mu_{2}t}{\sqrt{2t}}\right) - \operatorname{erf}\left(\frac{\vartheta_{1} + B - \mu_{2}t}{\sqrt{2t}}\right)\right] \right\},$$

which is equivalent to

$$\begin{split} P(t, x_2 \in [\vartheta_1, \vartheta_2]) &= \\ &= \frac{-A}{2\sqrt{2\pi t^3}} \exp\left[-\frac{(A + \mu_1 t)^2}{2t}\right] \left\{ \left[\operatorname{erf}\left(\frac{\vartheta_2 - B - \mu_2 t}{\sqrt{2t}}\right) - \operatorname{erf}\left(\frac{\vartheta_1 - B - \mu_2 t}{\sqrt{2t}}\right) \right] \right. \\ &\left. - \exp\left[-2B\mu_2\right] \left[\operatorname{erf}\left(\frac{\vartheta_2 + B - \mu_2 t}{\sqrt{2t}}\right) - \operatorname{erf}\left(\frac{\vartheta_1 + B - \mu_2 t}{\sqrt{2t}}\right) \right] \right\}, \end{split}$$

3.8 Case: $\rho = -0.5$ (PCRM and PCRMt)

If $\rho = -0.5$, it is $\sigma_u^2 = \sigma^2$ and $\sigma_v^2 = 3\sigma^2$, which gives

$$g_1(t, x_2) = \frac{\sigma^2}{2\sqrt{3}\pi\sigma^2 t^2} \sum_{i=0}^5 c_i \frac{1}{\sigma^2} \left(\frac{2}{3} \tilde{x}_{2,i} + \frac{4}{3} \tilde{x}_{1,i} \right) \exp\left[-\frac{(\tilde{x}_{1,i} + \tilde{x}_{2,i})^2 + \frac{1}{3} (\tilde{x}_{1,i} - \tilde{x}_{2,i})^2}{2\sigma^2 t} \right]$$
$$= \frac{1}{3\sqrt{3}\pi\sigma^2 t^2} \sum_{i=0}^5 c_i \exp\left[-\frac{\tilde{x}_{1,i}^2}{2\sigma^2 t} \right] (\tilde{x}_{2,i} + 2\tilde{x}_{1,i}) \exp\left[-\frac{(\tilde{x}_{2,i} + \frac{1}{2}\tilde{x}_{1,i})^2}{\frac{3}{4}2\sigma^2 t} \right].$$

Integrating the right hand side with respect to $\tilde{x}_{2,i}$ for each i using a variable transformation and adapted bounds $\tilde{\vartheta}_1, \tilde{\vartheta}_2$, we attain

$$\begin{split} P(t,x_2 \in [\vartheta_1,\vartheta_2]) &= \frac{1}{3\sqrt{3}\pi\sigma^2t^2} \sum_{i=0}^5 c_i \exp\left[-\frac{\tilde{x}_{1,i}^2}{2\sigma^2t}\right] \int_{\tilde{\vartheta}_1}^{\tilde{\vartheta}_2} \left(\tilde{x}_{2,i} + 2\tilde{x}_{1,i}\right) \exp\left[-\frac{(\tilde{x}_{2,i} + \frac{1}{2}\tilde{x}_{1,i})^2}{\frac{3}{4}2\sigma^2t}\right] d\tilde{x}_{2,i} \\ &= \frac{1}{3\sqrt{3}\pi\sigma^2t^2} \sum_{i=0}^5 c_i \exp\left[-\frac{\tilde{x}_{1,i}^2}{2\sigma^2t}\right] \\ &\qquad \times \left[\sqrt{\frac{\pi_4^3\sigma^2t}{2}} \left(2\tilde{x}_{1,i} + \frac{1}{2}\tilde{x}_{1,i}\right) \operatorname{erf}\left(\frac{\tilde{x}_{2,i} + \frac{1}{2}\tilde{x}_{1,i}}{\sqrt{\frac{3}{4}2\sigma^2t}}\right) - \frac{3}{4}\sigma^2t \exp\left[-\frac{(\tilde{x}_{2,i} + \frac{1}{2}\tilde{x}_{1,i})^2}{\frac{3}{4}2\sigma^2t}\right]\right]_{\tilde{\vartheta}_1}^{\tilde{\vartheta}_2} \\ &= \frac{1}{4\sqrt{3}\pi t} \sum_{i=0}^5 c_i \exp\left[-\frac{\tilde{x}_{1,i}^2}{2\sigma^2t}\right] \left[\sqrt{\frac{3\pi}{2\sigma^2t}}\tilde{x}_{1,i} \left(\operatorname{erf}\left(\frac{2\tilde{\vartheta}_2 + \tilde{x}_{1,i}}{\sqrt{6\sigma^2t}}\right) - \operatorname{erf}\left(\frac{2\tilde{\vartheta}_1 + \tilde{x}_{1,i}}{\sqrt{6\sigma^2t}}\right)\right) - \left(\exp\left[-\frac{(2\tilde{\vartheta}_2 + \tilde{x}_{1,i})^2}{6\sigma^2t}\right] - \exp\left[-\frac{(2\tilde{\vartheta}_1 + \tilde{x}_{1,i})^2}{6\sigma^2t}\right]\right)\right] \\ &= \frac{1}{4\sqrt{3}\pi t} \sum_{i=0}^5 c_i \exp\left[-\frac{\tilde{x}_{1,i}^2}{2\sigma^2t}\right] \\ &\qquad \times \left[\sqrt{\frac{3\pi}{2\sigma^2t}}\tilde{x}_{1,i} \left(\operatorname{erf}\left(\frac{2(\vartheta_2 - x_{i,2} - \mu_2 t) + \tilde{x}_{1,i}}{\sqrt{6\sigma^2t}}\right) - \operatorname{erf}\left(\frac{2(\vartheta_1 - x_{i,2} - \mu_2 t) + \tilde{x}_{1,i}}{\sqrt{6\sigma^2t}}\right)\right) - \left(\exp\left[-\frac{(2(\vartheta_2 - x_{i,2} - \mu_2 t) + \tilde{x}_{1,i}}{6\sigma^2t}\right] - \exp\left[-\frac{(2(\vartheta_1 - x_{i,2} - \mu_2 t) + \tilde{x}_{1,i}}{6\sigma^2t}\right]\right)\right]. \end{split}$$

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