Online Supplemental Material

A computational theory for the production of limb movements

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Here, we present the mathematical background necessary to understand the proposed model.

Optimal control with terminal constraints

We consider a dynamical system

$$\dot{\boldsymbol{x}}(t) = f\left[\boldsymbol{x}(t), \boldsymbol{u}(t)\right] \tag{S1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state of the system and $\boldsymbol{u} \in \mathbb{R}^m$ a control vector. An optimal control problem for this system is to find the control vector $\boldsymbol{u}(t)$ for $t \in [t_0; t_f]$ to minimize a performance index

$$J = \int_{t_0}^{t_f} L\left[\boldsymbol{x}(t), \boldsymbol{u}(t)\right]\, dt \tag{S2}$$

with boundary conditions

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0 \qquad \psi\left[\boldsymbol{x}(t_f)\right] = 0. \tag{S3}$$

This problem is the generic formulation corresponding to Equations 1,2,3 of the article.

Mayer formulation

We first show that the optimal control problem defined by Eq. S1, Eq. S2 and Eq. S3 can be equivalently written

$$\dot{\tilde{\boldsymbol{x}}}(t) = \tilde{f}\left[\tilde{\boldsymbol{x}}(t), \boldsymbol{u}(t)\right] \tag{S4}$$

$$\tilde{J} = \phi\left[\tilde{\boldsymbol{x}}(t_f)\right] \tag{S5}$$

$$\tilde{\boldsymbol{x}}(t_0) = \tilde{\boldsymbol{x}}_0 \qquad \tilde{\psi}\left[\tilde{\boldsymbol{x}}(t_f)\right] = 0 \tag{S6}$$

which is called the Mayer formulation and which is simpler for numerical methods.

We consider the supplementary state variable z defined by

$$\dot{z}(t) = L\left[\boldsymbol{x}(t), \boldsymbol{u}(t)\right]$$

and $z(t_0) = 0$. Thus $J = z(t_f)$. We define the new state variable

$$\tilde{\boldsymbol{x}}(t) = \begin{pmatrix} z(t) \\ \boldsymbol{x}(t) \end{pmatrix}.$$

We can reformulate the optimal control problem in the following way: find the control vector $\boldsymbol{u}(t)$ to minimize

$$\tilde{J} = \phi\left[\tilde{\boldsymbol{x}}(t_f)\right] = z(t_f) \tag{S7}$$

subject to

$$\dot{\tilde{\boldsymbol{x}}}(t) = \tilde{f}\left[\tilde{\boldsymbol{x}}(t), \boldsymbol{u}(t)\right] = \begin{pmatrix} L\left[\boldsymbol{x}(t), \boldsymbol{u}(t)\right] \\ f\left[\boldsymbol{x}(t), \boldsymbol{u}(t)\right] \end{pmatrix} \tag{S8}$$

and

$$\tilde{\boldsymbol{x}}(t_0) = \tilde{\boldsymbol{x}}_0 = \begin{pmatrix} 0 \\ \boldsymbol{x}_0 \end{pmatrix} \qquad \tilde{\psi}\left[\tilde{\boldsymbol{x}}(t_f)\right] = \begin{pmatrix} 0 \\ \psi\left[\boldsymbol{x}(t_f)\right] \end{pmatrix} = 0. \tag{S9}$$

Thus we can remove the integral term in the performance index. In the following we consider the problem defined by Eq. S4, Eq. S5 and Eq. S6. For simplicity, we remove the tilde sign.

Solution

We adjoin the constraints to the performance index with Lagrange multipliers $\boldsymbol{\nu}$ and $\boldsymbol{\lambda}(t)$

$$\bar{J} = \phi + \boldsymbol{\nu}^T \psi + \int_{t_0}^{t_f} \boldsymbol{\lambda}^T(t) \left\{ f\left[\boldsymbol{x}(t), \boldsymbol{u}(t)\right] - \dot{\boldsymbol{x}}(t) \right\} \, dt.$$

The Hamiltonian function is

$$\mathcal{H}\left[\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\lambda}(t)\right] = \mathcal{H}(t) = \boldsymbol{\lambda}^T(t) f[\boldsymbol{x}(t), \boldsymbol{u}(t)]. \tag{S10}$$

The generalized performance index can be written

$$\bar{J} = \Phi[\boldsymbol{x}(t_f)] - \boldsymbol{\lambda}^T(t_f)\boldsymbol{x}(t_f) + \boldsymbol{\lambda}^T(t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t_f} \left\{\mathcal{H}(t) + \dot{\boldsymbol{\lambda}}^T(t)\boldsymbol{x}(t)\right\}\, dt$$

following integration of the $\boldsymbol{\lambda}^T \dot{\boldsymbol{x}}$ by parts, where

$$\Phi = \phi + \boldsymbol{\nu}^T \psi \tag{S11}$$

A variation of $\bar{J}$ writes

$$\delta\bar{J} = \left[(\Phi_x - \boldsymbol{\lambda}^T)\delta x\right]_{t=t_f} + \left[\lambda^T \delta x\right]_{t=t_0} + \int_{t_0}^{t_f} \left[\left(\mathcal{H}_x + \dot{\boldsymbol{\lambda}}^T\right)\delta x + \mathcal{H}_u \delta u\right]\, dt$$

for variations $\delta x(t)$ and $\delta u(t)$. The Lagrange mutlipliers are chosen so that the coefficients of $\delta x(t)$ and $\delta x(t_f)$ vanish

$$\dot{\boldsymbol{\lambda}}^T = -\mathcal{H}_x = -\boldsymbol{\lambda}^T f_x, \tag{S12}$$

with boundary conditions

$$\boldsymbol{\lambda}^T(t_f) = \phi_x(t_f) + \boldsymbol{\nu}^T \psi_x(t_f). \tag{S13}$$

For a stationary solution, $\delta\bar{J} = 0$ for arbitrary $\delta u(t)$, which implies

$$\mathcal{H}_u = \boldsymbol{\lambda}^T f_u = 0 \qquad t_0 \leq t \leq t_f. \tag{S14}$$

The problem defined by Eq. S1, Eq. S12, Eq. S13 and Eq. S14 is a two-point boundary value problem which can be solved by classical integration methods (Bryson 1999).

Linear case

In the linear case, the problem is a first-order linear dynamical system which can be solved explicitly. The solution consists in a $2n \times 2n$ matrix $\mathbf{D}(t)$ such that

$$\begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{\lambda}(t) \end{pmatrix} = \mathbf{D}(t)\mathcal{C} \tag{S15}$$

is the solution at time t, where $\mathcal{C} \in \mathbb{R}^{2n}$ is a vector determined by the boundary conditions (Eq. S3). To simplify we use $\psi\left[\boldsymbol{x}(t_f)\right] = \boldsymbol{x}(t_f) - \boldsymbol{x}_f$, but more complex boundary conditions can be handled as well (see below). To obtain $\mathcal{C}$, we write

$$\begin{pmatrix} \boldsymbol{x}_0 \\ \boldsymbol{\lambda}(t_0) \end{pmatrix} = \mathbf{D}(t_0)\mathcal{C} = \begin{pmatrix} \mathbf{D}_{11}(t_0) & \mathbf{D}_{12}(t_0) \\ \mathbf{D}_{21}(t_0) & \mathbf{D}_{22}(t_0) \end{pmatrix} \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix}$$

and

$$\begin{pmatrix} \boldsymbol{x}_f \\ \boldsymbol{\lambda}(t_f) \end{pmatrix} = \mathbf{D}(t_f)\mathcal{C} = \begin{pmatrix} \mathbf{D}_{11}(t_f) & \mathbf{D}_{12}(t_f) \\ \mathbf{D}_{21}(t_f) & \mathbf{D}_{22}(t_f) \end{pmatrix} \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} \mathbf{D}_{11}(t_0) & \mathbf{D}_{12}(t_0) \\ \mathbf{D}_{11}(t_f) & \mathbf{D}_{12}(t_f) \end{pmatrix} \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x}_f \end{pmatrix},$$

which gives

$$\mathcal{C} = \begin{pmatrix} \mathbf{D}_{11}(t_0) & \mathbf{D}_{12}(t_0) \\ \mathbf{D}_{11}(t_f) & \mathbf{D}_{12}(t_f) \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x}_f \end{pmatrix}. \tag{S16}$$

Complete treatement of a linear case

Here we consider the problem of controlling an inertial point actuated by a linear muscle with a quadratic cost function. In Mayer formulation, the problem can be written

$$\begin{cases} \dot{x}_1 = u^2/2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_4/m \\ \dot{x}_4 = (-x_4 + x_5)/\tau \\ \dot{x}_5 = (-x_5 + u)/\tau \end{cases}$$

The Hamiltonian (Eq. S10) writes

$$\begin{aligned} \mathcal{H} \;=\; & \lambda_1 u^2/2 + \lambda_2 x_3 + \lambda_3 x_4/m + \\ & \lambda_4(-x_4 + x_5)/\tau + \lambda_5(-x_5 + u)/\tau \end{aligned}$$

from which the adjoint system (Eq. S12) can be obtained

$$
\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial \mathcal{H}}{\partial x_1} = 0 \\
\dot{\lambda}_2 &= -\frac{\partial \mathcal{H}}{\partial x_2} = 0 \\
\dot{\lambda}_3 &= -\frac{\partial \mathcal{H}}{\partial x_3} = -\lambda_2 \\
\dot{\lambda}_4 &= -\frac{\partial \mathcal{H}}{\partial x_4} = -\lambda_3/m + \lambda_4/\tau \\
\dot{\lambda}_5 &= -\frac{\partial \mathcal{H}}{\partial x_5} = -\lambda_4/\tau + \lambda_5/\tau
\end{aligned}
$$

The transversal condition (Eq. S14) is

$$
\mathcal{H}_u = \lambda_1 u + \lambda_5/\tau = 0,
$$

where λ_1 is a constant set at 1. The corresponding boundary value problem is

$$
\begin{cases}
\dot{x}_2 = x_3 \\
\dot{x}_3 = x_4/m \\
\dot{x}_4 = (-x_4 + x_5)/\tau \\
\dot{x}_5 = (-x_5 - \lambda_5/\tau)/\tau \\
\dot{\lambda}_2 = 0 \\
\dot{\lambda}_3 = -\lambda_2 \\
\dot{\lambda}_4 = -\lambda_3/m + \lambda_4/\tau \\
\dot{\lambda}_5 = -\lambda_4/\tau + \lambda_5/\tau
\end{cases}
\tag{S17}
$$

The constraints are defined by function Φ (Eq. S11) which can take different forms:

- Full constraints: position, velocity, activation, excitation

$$
\begin{aligned}
\Phi &= x_1(t_f) + \nu_2 \left[x_2(t_f) - x_2^f\right] + \nu_3 \left[x_3(t_f) - x_3^f\right] + \\
&\quad \nu_4 \left[x_4(t_f) - x_4^f\right] + \nu_5 \left[x_5(t_f) - x_5^f\right]
\end{aligned}
$$

- Partial constraints: position, velocity, activation

$$
\begin{aligned}
\Phi &= x_1(t_f) + \nu_2 \left[x_2(t_f) - x_2^f\right] + \nu_3 \left[x_3(t_f) - x_3^f\right] + \\
&\quad \nu_4 \left[x_4(t_f) - x_4^f\right]
\end{aligned}
$$

- Partial constraints: position, velocity

$$\Phi = x_1(t_f) + \nu_2 \left[x_2(t_f) - x_2^f\right] + \nu_3 \left[x_3(t_f) - x_3^f\right]$$

- Partial constraints: position

$$\Phi = x_1(t_f) + \nu_2 \left[x_2(t_f) - x_2^f\right]$$

The initial boundary conditions are

$$x_2(t_0) = x_2^0 \quad x_3(t_0) = x_3^0 \quad x_4(t_0) = x_4^0 \quad x_5(t_0) = x_5^0 \tag{S18}$$

The final boundary conditions are obtained using Eq. S13:

- Full constraints: position, velocity, activation, excitation

$$x_2(t_f) = x_2^f \quad x_3(t_f) = x_3^f \quad x_4(t_f) = x_4^f \quad x_5(t_f) = x_5^f \tag{S19}$$

$$\lambda_1(t_f) = 1 \quad \lambda_2(t_f) = \nu_2 \quad \lambda_3(t_f) = \nu_3 \quad \lambda_4(t_f) = \nu_4 \quad \lambda_5(t_f) = \nu_5 \tag{S20}$$

- Partial constraints: position, velocity, activation

$$x_2(t_f) = x_2^f \quad x_3(t_f) = x_3^f \quad x_4(t_f) = x_4^f \tag{S21}$$

$$\lambda_1(t_f) = 1 \quad \lambda_2(t_f) = \nu_2 \quad \lambda_3(t_f) = \nu_3 \quad \lambda_4(t_f) = \nu_4 \quad \lambda_5(t_f) = 0 \tag{S22}$$

- Partial constraints: position, velocity

$$x_2(t_f) = x_2^f \quad x_3(t_f) = x_3^f \tag{S23}$$

$$\lambda_1(t_f) = 1 \quad \lambda_2(t_f) = \nu_2 \quad \lambda_3(t_f) = \nu_3 \quad \lambda_4(t_f) = 0 \quad \lambda_5(t_f) = 0 \tag{S24}$$

- Partial constraints: position

$$x_2(t_f) = x_2^f \tag{S25}$$

$$\lambda_1(t_f) = 1 \quad \lambda_2(t_f) = \nu_2 \quad \lambda_3(t_f) = 0 \quad \lambda_4(t_f) = 0 \quad \lambda_5(t_f) = 0 \tag{S26}$$

From Eq. S27, the solution consists in a $2n \times 2n$ matrix $\mathbf{D}(t)$ ($n = 4$) such that

$$\begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{\lambda}(t) \end{pmatrix} = \mathbf{D}(t)\mathcal{C} \tag{S27}$$

is the solution at time t, where $\mathcal{C} \in \mathbb{R}^{2n}$ is a vector determined by the initial and final boundary conditions. Here $\mathbf{D}$ is the solution to the boundary value problem (Eq. S17), which can be obtained explicitly using tools of symbolic calculus.

To obtain $\mathcal{C}$, we write

$$\begin{pmatrix} \boldsymbol{x}_0 \\ \boldsymbol{\lambda}(t_0) \end{pmatrix} = \mathbf{D}(t_0)\mathcal{C} = \mathbf{D}^0\mathcal{C}$$

and

$$\begin{pmatrix} \boldsymbol{x}_f \\ \boldsymbol{\lambda}(t_f) \end{pmatrix} = \mathbf{D}(t_f)\mathcal{C} = \mathbf{D}^f\mathcal{C}$$

and we extract what is known from these relationships in the different cases (full constraints: Eq. S19 and Eq. S20; partial constraints on position, velocity, activation: Eq. S21 and Eq. S22; partial constraints on position, velocity: Eq. S23 and Eq. S24; partial constraints on position: Eq. S25 and Eq. S26).

We obtain a relationship

$$\mathrm{M}\boldsymbol{q} = \boldsymbol{p} \tag{S28}$$

where M contains elements of $\mathbf{D}^0$ and $\mathbf{D}^f$, $\boldsymbol{q}$ the vector $\mathcal{C}$ and some elements of $\boldsymbol{\nu}$, and $\boldsymbol{p}$ the vector $\boldsymbol{x}_0$ and some elements of $\boldsymbol{x}_f$. Taking $\boldsymbol{q} = \mathrm{M}^{-1}\boldsymbol{p}$ gives the vector $\mathcal{C}$.

For the case of full constraints (position, velocity, activation, excitation), there are 8 unknowns (8 in $\mathcal{C}$). We get 4 equations for $\boldsymbol{x}_0$ (Eq. S18), and 4 equations for $\boldsymbol{x}_f$ (Eq. S19), and Eq. S28 becomes

$$\begin{pmatrix}
\mathbf{D}^0_{11} & \mathbf{D}^0_{12} & \mathbf{D}^0_{13} & \mathbf{D}^0_{14} & \mathbf{D}^0_{15} & \mathbf{D}^0_{16} & \mathbf{D}^0_{17} & \mathbf{D}^0_{18} \\
\mathbf{D}^0_{21} & \mathbf{D}^0_{22} & \mathbf{D}^0_{23} & \mathbf{D}^0_{24} & \mathbf{D}^0_{25} & \mathbf{D}^0_{26} & \mathbf{D}^0_{27} & \mathbf{D}^0_{28} \\
\mathbf{D}^0_{31} & \mathbf{D}^0_{32} & \mathbf{D}^0_{33} & \mathbf{D}^0_{34} & \mathbf{D}^0_{35} & \mathbf{D}^0_{36} & \mathbf{D}^0_{37} & \mathbf{D}^0_{38} \\
\mathbf{D}^0_{41} & \mathbf{D}^0_{42} & \mathbf{D}^0_{43} & \mathbf{D}^0_{44} & \mathbf{D}^0_{45} & \mathbf{D}^0_{46} & \mathbf{D}^0_{47} & \mathbf{D}^0_{48} \\
\mathbf{D}^f_{11} & \mathbf{D}^f_{12} & \mathbf{D}^f_{13} & \mathbf{D}^f_{14} & \mathbf{D}^f_{15} & \mathbf{D}^f_{16} & \mathbf{D}^f_{17} & \mathbf{D}^f_{18} \\
\mathbf{D}^f_{21} & \mathbf{D}^f_{22} & \mathbf{D}^f_{23} & \mathbf{D}^f_{24} & \mathbf{D}^f_{25} & \mathbf{D}^f_{26} & \mathbf{D}^f_{27} & \mathbf{D}^f_{28} \\
\mathbf{D}^f_{31} & \mathbf{D}^f_{32} & \mathbf{D}^f_{33} & \mathbf{D}^f_{34} & \mathbf{D}^f_{35} & \mathbf{D}^f_{36} & \mathbf{D}^f_{37} & \mathbf{D}^f_{38} \\
\mathbf{D}^f_{41} & \mathbf{D}^f_{42} & \mathbf{D}^f_{43} & \mathbf{D}^f_{44} & \mathbf{D}^f_{45} & \mathbf{D}^f_{46} & \mathbf{D}^f_{47} & \mathbf{D}^f_{48}
\end{pmatrix}
\begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \\ \mathcal{C}_3 \\ \mathcal{C}_4 \\ \mathcal{C}_5 \\ \mathcal{C}_6 \\ \mathcal{C}_7 \\ \mathcal{C}_8 \end{pmatrix}
=
\begin{pmatrix} x^0_2 \\ x^0_3 \\ x^0_4 \\ x^0_5 \\ x^f_2 \\ x^f_3 \\ x^f_4 \\ x^f_5 \end{pmatrix}$$

For the case of partial constraints on position, velocity, and activation, there are 11 unknowns (8 in $\mathcal{C}$, ν_2, ν_3, ν_4). We get 4 equations for $\boldsymbol{x}_0$ (Eq. S18), 3 equations for $\boldsymbol{x}_f$ (Eq. S21), 4 equations for $\boldsymbol{\lambda}(t_f)$ (Eq. S22), and Eq. S28 becomes

$$
\begin{pmatrix}
\mathbf{D}^0_{11} & \mathbf{D}^0_{12} & \mathbf{D}^0_{13} & \mathbf{D}^0_{14} & \mathbf{D}^0_{15} & \mathbf{D}^0_{16} & \mathbf{D}^0_{17} & \mathbf{D}^0_{18} & 0 & 0 & 0 \\
\mathbf{D}^0_{21} & \mathbf{D}^0_{22} & \mathbf{D}^0_{23} & \mathbf{D}^0_{24} & \mathbf{D}^0_{25} & \mathbf{D}^0_{26} & \mathbf{D}^0_{27} & \mathbf{D}^0_{28} & 0 & 0 & 0 \\
\mathbf{D}^0_{31} & \mathbf{D}^0_{32} & \mathbf{D}^0_{33} & \mathbf{D}^0_{34} & \mathbf{D}^0_{35} & \mathbf{D}^0_{36} & \mathbf{D}^0_{37} & \mathbf{D}^0_{38} & 0 & 0 & 0 \\
\mathbf{D}^0_{41} & \mathbf{D}^0_{42} & \mathbf{D}^0_{43} & \mathbf{D}^0_{44} & \mathbf{D}^0_{45} & \mathbf{D}^0_{46} & \mathbf{D}^0_{47} & \mathbf{D}^0_{48} & 0 & 0 & 0 \\
\mathbf{D}^f_{11} & \mathbf{D}^f_{12} & \mathbf{D}^f_{13} & \mathbf{D}^f_{14} & \mathbf{D}^f_{15} & \mathbf{D}^f_{16} & \mathbf{D}^f_{17} & \mathbf{D}^f_{18} & 0 & 0 & 0 \\
\mathbf{D}^f_{21} & \mathbf{D}^f_{22} & \mathbf{D}^f_{23} & \mathbf{D}^f_{24} & \mathbf{D}^f_{25} & \mathbf{D}^f_{26} & \mathbf{D}^f_{27} & \mathbf{D}^f_{18} & 0 & 0 & 0 \\
\mathbf{D}^f_{31} & \mathbf{D}^f_{32} & \mathbf{D}^f_{33} & \mathbf{D}^f_{34} & \mathbf{D}^f_{35} & \mathbf{D}^f_{36} & \mathbf{D}^f_{37} & \mathbf{D}^f_{38} & 0 & 0 & 0 \\
\mathbf{D}^f_{51} & \mathbf{D}^f_{52} & \mathbf{D}^f_{53} & \mathbf{D}^f_{54} & \mathbf{D}^f_{55} & \mathbf{D}^f_{56} & \mathbf{D}^f_{57} & \mathbf{D}^f_{58} & -1 & 0 & 0 \\
\mathbf{D}^f_{61} & \mathbf{D}^f_{62} & \mathbf{D}^f_{63} & \mathbf{D}^f_{64} & \mathbf{D}^f_{65} & \mathbf{D}^f_{66} & \mathbf{D}^f_{67} & \mathbf{D}^f_{68} & 0 & -1 & 0 \\
\mathbf{D}^f_{71} & \mathbf{D}^f_{72} & \mathbf{D}^f_{73} & \mathbf{D}^f_{74} & \mathbf{D}^f_{75} & \mathbf{D}^f_{76} & \mathbf{D}^f_{77} & \mathbf{D}^f_{78} & 0 & 0 & -1 \\
\mathbf{D}^f_{81} & \mathbf{D}^f_{82} & \mathbf{D}^f_{83} & \mathbf{D}^f_{84} & \mathbf{D}^f_{85} & \mathbf{D}^f_{86} & \mathbf{D}^f_{87} & \mathbf{D}^f_{88} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \\ \mathcal{C}_3 \\ \mathcal{C}_4 \\ \mathcal{C}_5 \\ \mathcal{C}_6 \\ \mathcal{C}_7 \\ \mathcal{C}_8 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{pmatrix}
=
\begin{pmatrix} x^0_2 \\ x^0_3 \\ x^0_4 \\ x^0_5 \\ x^f_2 \\ x^f_3 \\ x^f_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

For the case of partial constraints on position and velocity, there are 10 unknowns (8 in $\mathcal{C}$, ν_2, ν_3). We get 4 equations for $\boldsymbol{x}_0$ (Eq. S18), 2 equations for $\boldsymbol{x}_f$ (Eq. S23), 4 equations for $\boldsymbol{\lambda}(t_f)$ (Eq. S24), and Eq. S28 becomes

$$
\begin{pmatrix}
\mathbf{D}^0_{11} & \mathbf{D}^0_{12} & \mathbf{D}^0_{13} & \mathbf{D}^0_{14} & \mathbf{D}^0_{15} & \mathbf{D}^0_{16} & \mathbf{D}^0_{17} & \mathbf{D}^0_{18} & 0 & 0 \\
\mathbf{D}^0_{21} & \mathbf{D}^0_{22} & \mathbf{D}^0_{23} & \mathbf{D}^0_{24} & \mathbf{D}^0_{25} & \mathbf{D}^0_{26} & \mathbf{D}^0_{27} & \mathbf{D}^0_{28} & 0 & 0 \\
\mathbf{D}^0_{31} & \mathbf{D}^0_{32} & \mathbf{D}^0_{33} & \mathbf{D}^0_{34} & \mathbf{D}^0_{35} & \mathbf{D}^0_{36} & \mathbf{D}^0_{37} & \mathbf{D}^0_{38} & 0 & 0 \\
\mathbf{D}^0_{41} & \mathbf{D}^0_{42} & \mathbf{D}^0_{43} & \mathbf{D}^0_{44} & \mathbf{D}^0_{45} & \mathbf{D}^0_{46} & \mathbf{D}^0_{47} & \mathbf{D}^0_{48} & 0 & 0 \\
\mathbf{D}^f_{11} & \mathbf{D}^f_{12} & \mathbf{D}^f_{13} & \mathbf{D}^f_{14} & \mathbf{D}^f_{15} & \mathbf{D}^f_{16} & \mathbf{D}^f_{17} & \mathbf{D}^f_{18} & 0 & 0 \\
\mathbf{D}^f_{21} & \mathbf{D}^f_{22} & \mathbf{D}^f_{23} & \mathbf{D}^f_{24} & \mathbf{D}^f_{25} & \mathbf{D}^f_{26} & \mathbf{D}^f_{27} & \mathbf{D}^f_{18} & 0 & 0 \\
\mathbf{D}^f_{51} & \mathbf{D}^f_{52} & \mathbf{D}^f_{53} & \mathbf{D}^f_{54} & \mathbf{D}^f_{55} & \mathbf{D}^f_{56} & \mathbf{D}^f_{57} & \mathbf{D}^f_{58} & -1 & 0 \\
\mathbf{D}^f_{61} & \mathbf{D}^f_{62} & \mathbf{D}^f_{63} & \mathbf{D}^f_{64} & \mathbf{D}^f_{65} & \mathbf{D}^f_{66} & \mathbf{D}^f_{67} & \mathbf{D}^f_{68} & 0 & -1 \\
\mathbf{D}^f_{71} & \mathbf{D}^f_{72} & \mathbf{D}^f_{73} & \mathbf{D}^f_{74} & \mathbf{D}^f_{75} & \mathbf{D}^f_{76} & \mathbf{D}^f_{77} & \mathbf{D}^f_{78} & 0 & 0 \\
\mathbf{D}^f_{81} & \mathbf{D}^f_{82} & \mathbf{D}^f_{83} & \mathbf{D}^f_{84} & \mathbf{D}^f_{85} & \mathbf{D}^f_{86} & \mathbf{D}^f_{87} & \mathbf{D}^f_{88} & 0 & 0
\end{pmatrix}
\begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \\ \mathcal{C}_3 \\ \mathcal{C}_4 \\ \mathcal{C}_5 \\ \mathcal{C}_6 \\ \mathcal{C}_7 \\ \mathcal{C}_8 \\ \nu_2 \\ \nu_3 \end{pmatrix}
=
\begin{pmatrix} x^0_2 \\ x^0_3 \\ x^0_4 \\ x^0_5 \\ x^f_2 \\ x^f_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

For the case of partial constraints on position, there are 9 unknowns (8 in $\mathcal{C}$, ν_2). We get 4 equations for $\boldsymbol{x}_0$ (Eq. S18), 1 equation for $\boldsymbol{x}_f$ (Eq. S25), 4 equations for $\boldsymbol{\lambda}(t_f)$ (Eq. S26), and Eq. S28 becomes

$$
\begin{pmatrix}
\mathbf{D}^0_{11} & \mathbf{D}^0_{12} & \mathbf{D}^0_{13} & \mathbf{D}^0_{14} & \mathbf{D}^0_{15} & \mathbf{D}^0_{16} & \mathbf{D}^0_{17} & \mathbf{D}^0_{18} & 0 \\
\mathbf{D}^0_{21} & \mathbf{D}^0_{22} & \mathbf{D}^0_{23} & \mathbf{D}^0_{24} & \mathbf{D}^0_{25} & \mathbf{D}^0_{26} & \mathbf{D}^0_{27} & \mathbf{D}^0_{28} & 0 \\
\mathbf{D}^0_{31} & \mathbf{D}^0_{32} & \mathbf{D}^0_{33} & \mathbf{D}^0_{34} & \mathbf{D}^0_{35} & \mathbf{D}^0_{36} & \mathbf{D}^0_{37} & \mathbf{D}^0_{38} & 0 \\
\mathbf{D}^0_{41} & \mathbf{D}^0_{42} & \mathbf{D}^0_{43} & \mathbf{D}^0_{44} & \mathbf{D}^0_{45} & \mathbf{D}^0_{46} & \mathbf{D}^0_{47} & \mathbf{D}^0_{48} & 0 \\
\mathbf{D}^f_{11} & \mathbf{D}^f_{12} & \mathbf{D}^f_{13} & \mathbf{D}^f_{14} & \mathbf{D}^f_{15} & \mathbf{D}^f_{16} & \mathbf{D}^f_{17} & \mathbf{D}^f_{18} & 0 \\
\mathbf{D}^f_{51} & \mathbf{D}^f_{52} & \mathbf{D}^f_{53} & \mathbf{D}^f_{54} & \mathbf{D}^f_{55} & \mathbf{D}^f_{56} & \mathbf{D}^f_{57} & \mathbf{D}^f_{58} & -1 \\
\mathbf{D}^f_{61} & \mathbf{D}^f_{62} & \mathbf{D}^f_{63} & \mathbf{D}^f_{64} & \mathbf{D}^f_{65} & \mathbf{D}^f_{66} & \mathbf{D}^f_{67} & \mathbf{D}^f_{68} & 0 \\
\mathbf{D}^f_{71} & \mathbf{D}^f_{72} & \mathbf{D}^f_{73} & \mathbf{D}^f_{74} & \mathbf{D}^f_{75} & \mathbf{D}^f_{76} & \mathbf{D}^f_{77} & \mathbf{D}^f_{78} & 0 \\
\mathbf{D}^f_{81} & \mathbf{D}^f_{82} & \mathbf{D}^f_{83} & \mathbf{D}^f_{84} & \mathbf{D}^f_{85} & \mathbf{D}^f_{86} & \mathbf{D}^f_{87} & \mathbf{D}^f_{88} & 0
\end{pmatrix}
\begin{pmatrix}
\mathcal{C}_1 \\ \mathcal{C}_2 \\ \mathcal{C}_3 \\ \mathcal{C}_4 \\ \mathcal{C}_5 \\ \mathcal{C}_6 \\ \mathcal{C}_7 \\ \mathcal{C}_8 \\ \nu_2
\end{pmatrix}
=
\begin{pmatrix}
x^0_2 \\ x^0_3 \\ x^0_4 \\ x^0_5 \\ x^f_2 \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
$$

References

Bryson A (1999) *Dynamic Optimization*. Englewood Cliffs, NJ: Prentice-Hall.