Supplementary Material: Efficient Alternatives for Bayesian Hypothesis Tests in Psychology

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The supplemental materials address several topics not covered in the main article. First, we provide analytic expressions for Bayes factors in one-sided tests for normal means and differences in normal means. Following this, we provide proofs of these theorems and those stated in the main article. Finally, we provide summaries of operating characteristics for z and two-sample t tests that show that these tests perform similarly to the one-sample t test discussed in the main article.

S1 Bayes factors for one-sided tests

S1. One-sample, one-sided, known variance test. Assume the conditions of [**1**] in the main article hold, except that now $H_1 : \mu \sim NM^+(0, \tau^2\sigma^2)$. Then the Bayes factor in favor of H_1 can be expressed as

$$\mathrm{BF}_{10}(\mathbf{x}) = 2(n\tau^2+1)^{-3/2}\left[(1+2w)e^w\left(1-\mathcal{N}(\sqrt{w/2})\right)+\sqrt{\frac{2w}{\pi}}\right], \tag{1}$$

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where

$$r = \frac{n\tau^2}{1+n\tau^2}, \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad Z = \sqrt{n}\bar{x}/\sigma, \quad \text{and} \quad w = rZ^2/2, \tag{2}$$

and $\mathcal{N}(x)$ is the standard normal distribution function.

S2. One-sample, one-sided, unknown variance test. Suppose the conditions in **[2]** of the main article hold, except now that σ^2 is unknown. Suppose further that the Jeffreys' prior density is assumed under both hypotheses. In this case, the closed-form expression for the Bayes factor is more complicated because it depends on the Gauss hypergeometric function, ${}_2F_1(a,b,c,x)$ and the beta function, $\mathrm{B}(a,b)$. These functions are available in many statistical and mathematical software packages, including **R** (see package "hypergeo" for ${}_2F_1$ (Hankin, 2016)). Using these functions, the Bayes factor in favor of the alternative hypothesis can be expressed as

$$\mathrm{BF}_{10}(\mathbf{x}) = \begin{cases} c_1 \Big[f_1 d_1^2 \left(1 - \mathcal{T}_{2\nu-1}\left(-d_1\sqrt{2\nu-1}\right)\right) + f_2\, d_1 |d_1|^{2(1-\nu)} + \\ \quad f_3\, |d_1|^{3-2\nu} \Big] & \text{if } \bar{x} < 0, \\ c_1 \mathrm{B}\left(3/2, \nu - 3/2\right) & \text{if } \bar{x} = 0, \\ c_1 \Big[f_1 d_1^2 \left(1 - \mathcal{T}_{2\nu-1}\left(-d_1\sqrt{2\nu-1}\right)\right) + f_2\, d_1 |d_1|^{2(1-\nu)} + \\ \quad f_3\, |d_1|^{3-2\nu} + 2 f_4 |d_1|^3 \Big] & \text{if } \bar{x} > 0, \end{cases} \tag{3}$$

where

$$c_1 = \frac{4\Gamma(\nu)}{\sqrt{\pi}(n\tau^2+1)^{3/2}\Gamma(n/2)}, \tag{4}$$

$$q = \frac{rn}{n-1}, \quad S = \sum_{i=1}^{n}(x_i - \bar{x})^2, \quad s^2 = \frac{S}{(n-1)}, \quad T = \frac{\sqrt{n}\bar{x}}{s}, \tag{5}$$

$$G = 1 + \frac{T^2}{n-1}, \quad \text{and} \quad H = 1 + \frac{(1-r)T^2}{(n-1)}. \tag{6}$$

Variables $\bar{x}$, S, r, T, G, and H are defined in (2, 5, 6), $\nu = (n+3)/2$, and $d_1 = \sqrt{r}T/\sqrt{(n-1)H}$. The variables $f_1 - f_4$ are defined as

$$f_1 = \mathrm{B}\left(\nu - 1/2, 1/2\right), \quad f_2 = \frac{{}_2F_1\left(\nu, \nu-1; \nu; -1/d_1^2\right)}{(\nu - 1)}, \tag{7}$$

$$f_3 = \frac{{}_2F_1\left(\nu, \nu - 3/2; \nu - 1/2; -1/d_1^2\right)}{(2\nu - 3)}, \quad f_4 = \frac{{}_2F_1\left(\nu, 3/2; 5/2; -d_1^2\right)}{3}. \tag{8}$$

The function $\mathcal{T}_{2\nu-1}(\cdot)$ denotes the cumulative distribution function of a Student t random variable on $(2\nu - 1)$ degrees of freedom.

S3. Two-sample, one-sided, known variance test. Assume the conditions in **[3]** of the main article hold, except that now $H_1 : \mu_2 - \mu_1 \sim NM^+(0, \tau^2\sigma^2)$. Then the Bayes factor in favor of H_1 can be expressed as

$$\mathrm{BF}_{10}(\mathbf{x}_1, \mathbf{x}_2) = 2\left(m\tau^2 + 1\right)^{-3/2}\left[e^w\left(1 + 2w\right)\left(1 - \mathcal{N}\left(\sqrt{w/2}\right)\right) + \sqrt{\frac{2w}{\pi}}\right], \tag{9}$$

where

$$\bar{x_i} = \sum_{j=1}^{n_i} x_{j,i}/n_i, \qquad n = n_1 + n_2, \qquad m = \frac{n_1 n_2}{n_1 + n_2}, \tag{10}$$

$$r = \frac{m\tau^2}{m\tau^2 + 1}, \qquad Z = \sqrt{m}(\bar{x_2} - \bar{x_1})/\sigma \qquad \text{and} \qquad w = \frac{rZ^2}{2}. \tag{11}$$

and $\mathcal{N}$ is again the standard normal distribution function.

S4. Two-sample, one-sided, unknown variance test. Suppose the conditions in **[4]** of the main article hold, except now that σ^2 is unknown. Suppose further that the Jeffreys' prior density for σ^2 is assumed under both hypotheses. Then the Bayes factor in favor of the alternative hypothesis can be expressed as

$$\mathrm{BF}_{10}(\mathbf{x}_1, \mathbf{x}_2) = \begin{cases} c_1 \Big[f_1 d_1^2 \left(1 - \mathcal{T}_{2\nu-1}\left(-d_1\sqrt{2\nu-1}\right)\right) + f_2\, d_1 |d_1|^{2(1-\nu)} + \\ \quad f_3\, |d_1|^{3-2\nu} \Big] & \text{if } \bar{x}_2 < \bar{x}_1, \\ c_1 \mathrm{B}\left(3/2, \nu - 3/2\right) & \text{if } \bar{x} = 0, \\ c_1 \Big[f_1 d_1^2 \left(1 - \mathcal{T}_{2\nu-1}\left(-d_1\sqrt{2\nu-1}\right)\right) + f_2\, d_1 |d_1|^{2(1-\nu)} + \\ \quad f_3\, |d_1|^{3-2\nu} + 2 f_4 |d_1|^3 \Big] & \text{if } \bar{x}_2 > \bar{x}_1, \end{cases} \tag{12}$$

where

$$c_1 = \frac{2\Gamma(\nu)}{\sqrt{\pi}(m\tau^2+1)^{3/2}\Gamma\left((n-1)/2\right)}, \tag{13}$$

$$T = \frac{\sqrt{m}(\bar{x}_1 - \bar{x}_2)}{\sqrt{S/(n-2)}}, \qquad G = 1 + \frac{T^2}{(n-2)}, \qquad H = 1 + \frac{(1-r)T^2}{(n-2)}. \tag{14}$$

and $\nu = n/2 + 1$. The variables $f_1 - f_4$ are as defined in (7,8), but with $d_1 = \sqrt{r}T/\sqrt{(n-2)H}$.

S2 Proofs of theorems for one-sample tests

S2.1 Variance known

S2.1.1 Two-sided tests

Suppose $\mathbf{x} = (x_1, \ldots, x_n)$ are i.i.d. observations from a $N(\mu, \sigma^2)$ distribution with σ^2 known. The null hypothesis specifies that $H_0 : \mu = 0$. Under H_1, we assume that μ is drawn from a normal moment prior density specified by

$$p_{NM}(\mu \,|\, \tau^2, \sigma^2) = \frac{1}{\sqrt{2\pi}\tau^3\sigma^3}\mu^2 \exp\left(-\frac{\mu^2}{2\tau^2\sigma^2}\right) \quad \text{for } \mu \in \mathbb{R}. \tag{15}$$

Theorem S2.1. *Under the null hypothesis $H_0 : \mu = 0$ and σ^2 known, the marginal density of $\mathbf{x}$ is given by*

$$m_0(\mathbf{x} \mid \sigma^2) = c \exp\left(-\frac{n\bar{x}^2}{2\sigma^2}\right), \tag{16}$$

where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad S = \sum_{i=1}^{n}(x_i - \bar{x})^2, \quad \textit{and} \quad c = (2\pi\sigma^2)^{-n/2}\exp\left(-\frac{S}{2\sigma^2}\right). \tag{17}$$

Proof: The marginal density under the data under the simple null hypothesis is simply the sampling density of the data. Thus,

$$m_0(\mathbf{x} \mid \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right) \tag{18}$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{S}{2\sigma^2} - \frac{n(\bar{x} - \mu)^2}{2\sigma^2}\right). \tag{19}$$

Noting $\mu = 0$ under H_0, the result follows.

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Theorem S2.2. *Under the alternative hypothesis H_1 that μ is drawn* a priori *from the normal moment prior (15) and σ^2 known, the marginal density of $\mathbf{x}$ is given by*

$$m_1(\mathbf{x} \mid \sigma^2) = \frac{c\, a^{3/2}}{\tau^3\sigma^2}\left(\sigma^2 + an^2\bar{x}^2\right)\exp\left[-\frac{an\bar{x}^2}{2\tau^2\sigma^2}\right], \tag{20}$$

where $a = 1/(n + \tau^{-2})$ and c is defined in (17).

Proof: Substituting the expression for the sampling density of the data obtained in the proof of Theorem S2.1, multiplying by the prior on μ, and integrating to obtain the marginal density leads to

$$m_1(\mathbf{x} \,|\, \sigma^2) = \int_{-\infty}^{\infty} \frac{c}{\sqrt{2\pi}\tau^3\sigma^3}\mu^2 \exp\left(-\frac{\mu^2}{2\tau^2\sigma^2}\right) \exp\left(-\frac{n(\bar{x}-\mu)^2}{2\sigma^2}\right) d\mu \tag{21}$$

$$= \int_{-\infty}^{\infty} \frac{c}{\sqrt{2\pi}\tau^3\sigma^3}\mu^2 \exp\left[-\frac{1}{2\sigma^2}\left(\frac{\mu^2}{\tau^2} + n(\bar{x}-\mu)^2\right)\right] d\mu. \tag{22}$$

Because

$$\frac{\mu^2}{\tau^2} + n(\bar{x}-\mu)^2 = \frac{1}{a}\left(\mu - an\bar{x}\right)^2 + \frac{an\bar{x}^2}{\tau^2}, \tag{23}$$

it follows that

$$m_1(\mathbf{x} \,|\, \sigma^2) = \int_{-\infty}^{\infty} \frac{c}{\sqrt{2\pi}\tau^3\sigma^3}\mu^2 \exp\left\{-\frac{1}{2\sigma^2}\left[\frac{1}{a}\left(\mu - an\bar{x}\right)^2 + n\bar{x}^2 - an^2\bar{x}^2\right]\right\} d\mu \tag{24}$$

$$= \frac{\sqrt{a}c}{\tau^3\sigma^2} \exp\left(-\frac{an\bar{x}^2}{2\tau^2\sigma^2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi a}\sigma}\mu^2 \exp\left[-\frac{(\mu - an\bar{x})^2}{2a\sigma^2}\right] d\mu. \tag{25}$$

The integral represents the second moment of a normal distribution with mean $an\bar{x}$ and variance $a\sigma^2$. Thus

$$m_1(\mathbf{x} \,|\, \sigma^2) = \frac{\sqrt{a}c}{\tau^3\sigma^2}\left[a\sigma^2 + (an\bar{x})^2\right] \exp\left(-\frac{an\bar{x}^2}{2\tau^2\sigma^2}\right). \tag{26}$$

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Theorem S2.3. *Under the assumptions of Thm S2.1 and S2.2, the Bayes factor in favor of the alternative hypothesis H_1 against the null hypothesis H_0 is given by*

$$BF_{10}(\mathbf{x} \,|\, \sigma^2) = (n\tau^2+1)^{-3/2}\left(1 + r\,T^2\right) \exp\left(\frac{r\,T^2}{2}\right), \tag{27}$$

where $r = n\tau^2/\left(n\tau^2+1\right)$ and $T = \sqrt{n}\bar{x}/\sigma$.

Proof: Following the definition of the Bayes factor and substituting the expression for the marginal density of $\mathbf{x}$ from Thm S2.1 and S2.2 leads to

$$\begin{aligned}
& \mathrm{BF}_{10}(\mathbf{x}\,|\,\sigma^2) && (28)\\
= \; & \frac{m_1(\mathbf{x}\,|\,\sigma^2)}{m_0(\mathbf{x}\,|\,\sigma^2)} && (29)\\
= \; & \frac{1}{\sigma^2(n\tau^2+1)^{3/2}}\left[\sigma^2+\frac{n\bar{x}^2}{1+(n\tau^2)^{-1}}\right]\exp\left(\frac{n^2\tau^2\bar{x}^2}{2\sigma^2(n\tau^2+1)}\right) && (30)\\
= \; & (n\tau^2+1)^{-3/2}\left(1+r\,T^2\right)\exp\left(\frac{r\,T^2}{2}\right). && (31)
\end{aligned}$$

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S2.1.2 One-sided tests

Assume the same setup as in Section S2.1.1, except that we now wish to test $H_0 : \mu = 0$ versus $H_1 : \mu > 0$. To this end, under H_1 we assume that μ is drawn from a normal moment prior truncated on $(0, \infty)$. The density is specified by

$$p_{NM}(\mu\,|\,\tau^2,\sigma^2) = \frac{\sqrt{2}}{\sqrt{\pi}\tau^3\sigma^3}\mu^2\exp\left(-\frac{\mu^2}{2\tau^2\sigma^2}\right) \quad \text{for } \mu > 0. \tag{32}$$

Under this setup we note that the marginal density of $\mathbf{x}$ under the null hypothesis $H_0 : \mu = 0$ is the same as in Theorem S2.1.

Theorem S2.4. *Under the alternative hypothesis H_1 that μ is drawn* a priori *from the normal moment prior (32) and σ^2 known, the marginal density $m_1(\mathbf{x}\,|\,\sigma^2)$ of $\mathbf{x}$ is given by*

$$\frac{c}{(n\tau^2+1)^{3/2}}\exp\left(-\frac{d^2}{n\tau^2}\right)\left[\left(2d^2+1\right)\{1-erf(-d)\}+\frac{2d}{\sqrt{\pi}}\exp\left(-d^2\right)\right], \tag{33}$$

where a is as in Theorem S2.2, c is as in Theorem S2.1, and $d = \sqrt{a}\,n\bar{x}/\sqrt{2}\,\sigma$.

Proof: Substituting the expression for the sampling density of the data obtained in the proof of Theorem S2.1, multiplying by the prior (32) on μ, and integrating to obtain the marginal density leads to

$$m_1(\mathbf{x} \mid \sigma^2) = \int_0^\infty \frac{c\sqrt{2}}{\sqrt{\pi}\tau^3\sigma^3}\mu^2 \exp\left(-\frac{\mu^2}{2\tau^2\sigma^2}\right) \exp\left(-\frac{n(\bar{x}-\mu)^2}{2\sigma^2}\right) d\mu. \tag{34}$$

Using the identity (23) and using 2.1.3.1 from Korotkov and Korotkov (2020) leads to

$$\begin{aligned}
m_1(\mathbf{x} \mid \sigma^2) &= \frac{c\sqrt{2}}{\sqrt{\pi}\tau^3\sigma^3} \exp\left(-\frac{an\bar{x}^2}{2\tau^2\sigma^2}\right) \int_0^\infty \mu^2 \exp\left[-\frac{(\mu - an\bar{x})^2}{2a\sigma^2}\right] d\mu && (35)\\
&= \frac{c\sqrt{2}}{\sqrt{\pi}\tau^3\sigma^3} \exp\left(-\frac{an\bar{x}^2}{2\tau^2\sigma^2}\right) \times && (36)\\
&\quad \left[\frac{\sqrt{\pi}a^{3/2}\sigma^3}{\sqrt{2}}\left(\frac{an^2\bar{x}^2}{\sigma^2}+1\right)\left\{1 - \text{erf}\left(-\frac{\sqrt{a}n\bar{x}}{\sqrt{2}\sigma}\right)\right\} + \right. && (37)\\
&\quad \left. a^2 n\bar{x}\sigma^2 \exp\left(-\frac{an^2\bar{x}^2}{2\sigma^2}\right)\right] && (38)\\
&= \frac{ca^{3/2}}{\tau^3} \exp\left(-\frac{an\bar{x}^2}{2\tau^2\sigma^2}\right) \times && (39)\\
&\quad \left[\left(\frac{an^2\bar{x}^2}{\sigma^2}+1\right)\left\{1 - \text{erf}\left(-\frac{\sqrt{a}n\bar{x}}{\sqrt{2}\sigma}\right)\right\} + \frac{\sqrt{2a}\,n\bar{x}}{\sqrt{\pi}\sigma} \exp\left(-\frac{an^2\bar{x}^2}{2\sigma^2}\right)\right] && (40)\\
&= \frac{c}{(n\tau^2+1)^{3/2}} \exp\left(-\frac{d^2}{n\tau^2}\right) \times && (41)\\
&\quad \left[\left(2d^2+1\right)\left\{1 - \text{erf}\left(-d\right)\right\} + \frac{2d}{\sqrt{\pi}} \exp\left(-d^2\right)\right]. && (42)
\end{aligned}$$

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Theorem S2.5. *Under the assumptions Section S2.1.2, the Bayes factor $BF_{10}(\mathbf{x} \mid \sigma^2)$ in favor of the alternative hypothesis H_1 against the null hypothesis H_0 is given by*

$$\left(n\tau^2+1\right)^{-3/2} \exp\left(\frac{rT^2}{2}\right)\left[\left(rT^2+1\right)\left(1 - erf\left(-\frac{\sqrt{r}T}{\sqrt{2}}\right)\right) + \frac{\sqrt{2r}T}{\sqrt{\pi}} \exp\left(-\frac{rT^2}{2}\right)\right], \tag{43}$$

where $r = n\tau^2/\left(n\tau^2+1\right)$ *and* $T = \sqrt{n}\bar{x}/\sigma$.

Proof: Following the definition of the Bayes factor and substituting the expression for the marginal density of $\mathbf{x}$ from Thm S2.1 and S2.4 leads to

$$
\begin{aligned}
& \mathrm{BF}_{10}(\mathbf{x}\,|\,\sigma^2) && (44)\\
= \; & \frac{m_1(\mathbf{x}\,|\,\sigma^2)}{m_0(\mathbf{x}\,|\,\sigma^2)} && (45)\\
= \; & \left(n\tau^2+1\right)^{-3/2} \exp\left(\frac{n^2\tau^2\bar{x}^2}{2\sigma^2(n\tau^2+1)}\right) \times && (46)\\
& \left[\left(\frac{n^2\tau^2\bar{x}^2}{\sigma^2(n\tau^2+1)}+1\right)\left(1-\mathrm{erf}\left(-\frac{n\tau\bar{x}}{\sigma\sqrt{2\left(n\tau^2+1\right)}}\right)\right)+\right. && (47)\\
& \left.\frac{\sqrt{2}n\tau\bar{x}}{\sigma\sqrt{\pi\left(n\tau^2+1\right)}} \exp\left(-\frac{n^2\tau^2\bar{x}^2}{2\sigma^2(n\tau^2+1)}\right)\right] && (48)\\
= \; & \left(n\tau^2+1\right)^{-3/2} \exp\left(\frac{rT^2}{2}\right) \times && (49)\\
& \left[\left(rT^2+1\right)\left(1-\mathrm{erf}\left(-\frac{\sqrt{r}T}{\sqrt{2}}\right)\right)+\frac{\sqrt{2r}T}{\sqrt{\pi}} \exp\left(-\frac{rT^2}{2}\right)\right]. && (50)
\end{aligned}
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S2.2 Variance unknown

S2.2.1 Two-sided tests

As in Section S2.1.1 let $\mathbf{x} = (x_1, \ldots, x_n)$ denote i.i.d. observations from a $N(\mu, \sigma^2)$ distribution, but assume now that σ^2 unknown. We again wish to test $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$. Under H_1, the prior on μ, given σ^2, is again specified as a normal moment prior (15). To complete the model specification, under both H_0 and H_1 we also assume an inverse gamma prior on σ^2, parameterized here as

$$\pi(\sigma^2 \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\alpha-1} \exp\left(- \frac{\beta}{\sigma^2} \right) \quad \text{for } \sigma^2 > 0, \tag{51}$$

with shape parameter $\alpha\,(> 0)$ and scale parameter $\beta\,(> 0)$.

Theorem S2.6. *Under these assumptions and assuming H_0 to be true, the marginal density of the data $m(\mathbf{x})$ is given by*

$$m_0(\mathbf{x}) = \frac{(2\pi)^{-n/2}\,\beta^\alpha\,\Gamma(n/2+\alpha)}{\Gamma(\alpha)} \left[\frac{S + n\bar{x}^2}{2} + \beta \right]^{-n/2-\alpha}, \tag{52}$$

where S is as defined in (17).

Proof: Since H_0 is a point null hypothesis, the prior on μ is a degenerate distribution with all the mass at μ_0. So the marginal density $m(\mathbf{x})$ can be expressed as

$$m_0(\mathbf{x}) = \int \pi(\sigma^2 \mid \alpha, \beta) \prod_{i=1}^{n} \phi(x_i \mid 0, \sigma^2)\, d\sigma^2 \tag{53}$$

$$= \frac{(2\pi)^{-n/2}\,\beta^\alpha}{\Gamma(\alpha)} \int (\sigma^2)^{-n/2-\alpha-1} \exp\left[- \frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2 - \frac{\beta}{\sigma^2} \right] d\sigma^2. \tag{54}$$

Noting that the above integral with respect to σ^2 is proportional to an Inverse-gamma density yields (52).

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Theorem S2.7. *Under the assumptions above and assuming H_1 is true, the marginal density of the data $m(\mathbf{x})$ can be expressed as*

$$m_1(\mathbf{x}) = c^* \left[\frac{S + d\bar{x}^2}{2} + \beta \right]^{-n/2-\alpha-1} \left[\frac{S + d\bar{x}^2}{2} + \beta + nd\tau^2\bar{x}^2 \left(\frac{n}{2} + \alpha \right) \right], \tag{55}$$

where $\bar{x}$, S are as in (17), and

$$c^* = \frac{(2\pi)^{-n/2}\,\beta^{\alpha}\,\Gamma(n/2+\alpha)}{(n\tau^2+1)^{3/2}\,\Gamma(\alpha)}, \quad \textit{and} \quad d = \frac{n}{n\tau^2+1}. \tag{56}$$

Proof: The marginal density $m_1(\mathbf{x})$, given τ^2, can be obtained by integrating (20) over the prior on σ^2, leading to

$$m_1(\mathbf{x}) = \int \pi(\sigma^2 \mid \alpha, \beta) \times \frac{c}{\sigma^2(n\tau^2+1)^{3/2}} \exp\left[-\frac{d\bar{x}^2}{2\sigma^2}\right]\left(\sigma^2 + nd\tau^2\bar{x}^2\right) d\sigma^2 \tag{57}$$

$$= \frac{(2\pi)^{-n/2}\beta^{\alpha}}{(n\tau^2+1)^{3/2}\,\Gamma(\alpha)} \int (\sigma^2)^{-n/2-\alpha-2} \exp\left[-\frac{(S+d\bar{x}^2+2\beta)}{2\sigma^2}\right]\left(\sigma^2 + nd\tau^2\bar{x}^2\right) d\sigma^2. \tag{58}$$

Noting that the integrals with respect to σ^2 are proportional to an Inverse-gamma density results in

$$m_1(\mathbf{x}) = \frac{(2\pi)^{-n/2}\beta^{\alpha}}{(n\tau^2+1)^{3/2}\,\Gamma(\alpha)}\left[\Gamma\left(\frac{n}{2}+\alpha\right)\left\{\frac{S}{2}+\frac{d\bar{x}^2}{2}+\beta\right\}^{-n/2-\alpha} + \right. \tag{59}$$

$$\left. nd\tau^2\bar{x}^2\,\Gamma\left(\frac{n}{2}+\alpha+1\right)\left\{\frac{S}{2}+\frac{d\bar{x}^2}{2}+\beta\right\}^{-n/2-\alpha-1}\right] \tag{60}$$

$$= \quad c^*\left[\frac{S+d\bar{x}^2}{2}+\beta\right]^{-n/2-\alpha-1}\left[\frac{S+d\bar{x}^2}{2}+\beta+nd\tau^2\bar{x}^2\left(\frac{n}{2}+\alpha\right)\right]. \tag{61}$$

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Theorem S2.8. *Under the assumptions of Thm S2.6 and S2.7, the Bayes factor in favor of the alternative hypothesis H_1 against the null hypothesis H_0 is given by*

$$BF_{10}(\mathbf{x}) = (n\tau^2+1)^{-3/2}\left(\frac{G}{H}\right)^{n/2+\alpha}\left(1+\frac{qT^2}{H}\right), \tag{62}$$

where

$$r = \frac{n\tau^2}{n\tau^2+1}, \qquad q = \frac{2r\,(n/2+\alpha)}{n-1}, \qquad T = \frac{\sqrt{n}\bar{x}}{\sqrt{S/(n-1)}}, \tag{63}$$

$$G = 1 + \frac{T^2}{n-1} + \frac{2\beta}{S}, \qquad H = 1 + \frac{(1-r)\,T^2}{n-1} + \frac{2\beta}{S}. \tag{64}$$

and S *is as in (17).*

Proof: Following the definition of the Bayes factor and substituting the expression for the marginal density of $\mathbf{x}$ from Thm S2.1 and S2.2 leads to

$$\begin{aligned}
& \mathrm{BF}_{10}(\mathbf{x}) && (65)\\
= \;& \frac{m_1(\mathbf{x})}{m_0(\mathbf{x})} && (66)\\
= \;& (n\tau^2+1)^{-3/2}\left[\frac{(S+n\bar{x}^2)/2+\beta}{(S+d\bar{x}^2)/2+\beta}\right]^{n/2+\alpha}\left[1+\frac{n^2\tau^2\bar{x}^2\,(n/2+\alpha)}{(n\tau^2+1)\,((S+d\bar{x}^2)/2+\beta)}\right] && (67)\\
= \;& (n\tau^2+1)^{-3/2}\left[\frac{1+T^2/(n-1)+2\beta/S}{1+T^2/\{(n\tau^2+1)(n-1)\}+2\beta/S}\right]^{n/2+\alpha}\times && (68)\\
& \left[1+\frac{2\tau^2 n(n/2+\alpha)}{(n\tau^2+1)}\frac{T^2/(n-1)}{1+T^2/\{(n\tau^2+1)(n-1)\}+2\beta/S}\right] && (69)\\
= \;& (n\tau^2+1)^{-3/2}\left(\frac{G}{H}\right)^{n/2+\alpha}\left(1+\frac{qT^2}{H}\right). && (70)
\end{aligned}$$

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S2.2.2 One-sided tests

Assume the same setup as in Section S2.2.1, except that we now wish to test $H_0 : \mu = 0$ versus $H_1 : \mu > 0$. As in Section S2.1.2, the prior on μ given σ^2 under H_1 is specified as (32), a normal moment prior truncated on $(0, \infty)$. To complete the model specification, under both H_0 and H_1 we assume an inverse gamma prior on σ^2 defined by (51). Under these assumptions and assuming H_0 to be true, the marginal density of the data $m_0(\mathbf{x})$ is the same as in Theorem S2.6.

Theorem S2.9. *Under the assumptions above and assuming H_1 is true, the marginal density of the data $m_1(\mathbf{x})$ can be expressed as*

$$\begin{cases} c^* \left(f_1 d^2 \left(1 - F_{2\nu-1}\left(-d\sqrt{2\nu-1}\right)\right) + f_2\, d\, |d|^{2(1-\nu)} + f_3\, |d|^{3-2\nu} \right) & \text{if } \bar{x} < 0, \\ c^* B\left(3/2, \nu - 3/2\right) & \text{if } \bar{x} = 0, \\ c^* \left(f_1 d^2 \left(1 - F_{2\nu-1}\left(-d\sqrt{2\nu-1}\right)\right) + f_2\, d\, |d|^{2(1-\nu)} + \right. & \\ \quad \left. f_3\, |d|^{3-2\nu} + 2 f_4\, |d|^3 \right) & \text{if } \bar{x} > 0, \end{cases} \tag{71}$$

where $\bar{x}$, S are as in (17), $d = \sqrt{a}\, n\bar{x}/\sqrt{2A_1}$, $\nu = (n+3)/2 + \alpha$, $a = 1/\left(n + \tau^{-2}\right)$,

$$c^* = \frac{(2\pi)^{-n/2}\, 4\beta^\alpha\, \Gamma(\nu)}{\sqrt{\pi}(n\tau^2+1)^{3/2}\, \Gamma(\alpha) A_1^{n/2+\alpha}}, \quad A_1 = \beta + \frac{S}{2} + \frac{an\bar{x}^2}{2\tau^2}, \tag{72}$$

$$f_1 = B\left(\nu - 1/2, 1/2\right), \quad f_2 = \frac{{}_2F_1\left(\nu, \nu-1; \nu; -1/d^2\right)}{(\nu - 1)}, \tag{73}$$

$$f_3 = \frac{{}_2F_1\left(\nu, \nu - 3/2; \nu - 1/2; -1/d^2\right)}{(2\nu - 3)}, \quad f_4 = \frac{{}_2F_1\left(\nu, 3/2; 5/2; -d^2\right)}{3}, \tag{74}$$

$B(\cdot,\cdot)$ is the Beta function, $F_{2\nu-1}$ is the cdf of the Student's t distribution (center 0 and scale 1) with degrees of freedom $2\nu - 1$, and ${}_2F_1$ is the Gauss hypergeometric function.

Proof: Substituting the expression for the sampling density of the data obtained in the proof of Theorem S2.1, multiplying by the priors on $\mu \,|\, \sigma^2$ and σ^2, and integrating to obtain the marginal density given τ^2 leads to

$$m_1(\mathbf{x}) = \int_0^\infty \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\alpha-1} \exp\left(-\frac{\beta}{\sigma^2}\right) \frac{\sqrt{2}}{\sqrt{\pi}\tau^3\sigma^3} \mu^2 \exp\left(-\frac{\mu^2}{2\tau^2\sigma^2}\right) \times \tag{75}$$

$$(2\pi\sigma^2)^{-n/2} \exp\left(-\frac{S}{2\sigma^2} - \frac{n(\bar{x}-\mu)^2}{2\sigma^2}\right) d\sigma^2\, d\mu \tag{76}$$

$$= \frac{\beta^\alpha \Gamma(\nu)}{2^{(n-1)/2}\,\pi^{(n+1)/2}\,\tau^3\,\Gamma(\alpha)} \int_0^\infty \mu^2 \left[\beta + \frac{S}{2} + \frac{1}{2}\left(\frac{\mu^2}{\tau^2} + n(\bar{x}-\mu)^2\right)\right]^{-\nu} d\mu. \tag{77}$$

where $\nu = (n+3)/2 + \alpha$. Define $A_1 = \beta + S/2 + an\bar{x}^2/2\tau^2$ and

$$A_2 = \frac{\beta^\alpha \Gamma(\nu)}{2^{(n-1)/2}\,\pi^{(n+1)/2}\,\tau^3\,\Gamma(\alpha)\,A_1^\nu}. \tag{78}$$

Using the identity (23) and some algebraic simplifications lead to

$$m_1(\mathbf{x}) = A_2 \int_0^\infty \mu^2 \left(1 + \frac{(\mu - an\bar{x})^2}{2aA_1}\right)^{-\nu} d\mu \tag{79}$$

$$= A_2 \int_{-an\bar{x}}^\infty (u + an\bar{x})^2 \left(1 + \frac{u^2}{2aA_1}\right)^{-\nu} du \tag{80}$$

$$= A_2 \left(a^2 n^2 \bar{x}^2\, I_0(-an\bar{x}) + 2an\bar{x}\, I_1(-an\bar{x}) + I_2(-an\bar{x})\right) \tag{81}$$

$$= A_2 \left(m_{10} + m_{11} + m_{12}\right), \tag{82}$$

where

$$I_k(g) = \int_g^\infty u^k \left(1 + \frac{u^2}{2aA_1}\right)^{-\nu} du \quad \text{for } g \in \mathbb{R}, k \geq 0, \tag{83}$$

$$m_{10} = a^2 n^2 \bar{x}^2\, I_0(-an\bar{x}), \quad m_{11} = 2an\bar{x}\, I_1(-an\bar{x}), \quad m_{12} = I_2(-an\bar{x}). \tag{84}$$

For $I_0(-an\bar{x})$, first doing a change of variable with $w/\sqrt{2\nu-1} = u/\sqrt{2aA_1}$ and then some algebraic simplifications lead to

$$
\begin{aligned}
I_0\left(-an\bar{x}\right) &= \int_{-an\bar{x}}^{\infty} \left(1+\frac{u^2}{2aA_1}\right)^{-\nu} du & (85)\\
&= \left(\frac{2aA_1}{2\nu-1}\right)^{1/2} \int_{-n\bar{x}\sqrt{\frac{(2\nu-1)a}{2A_1}}}^{\infty} \left(1+\frac{w^2}{2\nu-1}\right)^{-((2\nu-1)+1)/2} dw & (86)\\
&= \sqrt{2aA_1}\,\mathrm{B}\left((2\nu-1)/2, 1/2\right) \left[1-F_{2\nu-1}\left(-n\bar{x}\sqrt{\frac{(2\nu-1)a}{2A_1}}\right)\right], & (87)
\end{aligned}
$$

where $\mathrm{B}(\cdot,\cdot)$ is the Beta function and $F_{2\nu-1}$ is the cdf of the Student's t distribution (center 0 and scale 1) with degrees of freedom $2\nu-1$. Following this we get

$$
m_{10} = a^2 n^2 \bar{x}^2\, I_0\left(-an\bar{x}\right) = \left(2aA_1\right)^{3/2} f_1 d^2 \left(1-F_{2\nu-1}\left(-d\sqrt{2\nu-1}\right)\right), \qquad (88)
$$

where $d = \sqrt{a}\, n\bar{x}/\sqrt{2A_1}$ and $f_1 = \mathrm{B}\left(\nu-1/2, 1/2\right)$. For $I_1(-an\bar{x})$ and $I_2(-an\bar{x})$, we note that for integers k,

$$
I_k(g) = \begin{cases} I_k\left(|g|\right) & \text{if } g \geq 0\text{, or } g<0 \text{ and } k \text{ is odd,} \\ I_k\left(|g|\right) + 2J_k\left(|g|\right) & \text{if } g<0 \text{ and } k \text{ is even,} \end{cases} \qquad (89)
$$

where for $g > 0$,

$$
\begin{aligned}
I_k(g) &= \int_{g}^{\infty} u^k \left(1+\frac{u^2}{2aA_1}\right)^{-\nu} du = \frac{1}{2}\int_{g^2}^{\infty} w^{(k+1)/2-1} \left(1+\frac{w}{2aA_1}\right)^{-\nu} dw, & (90)\\
J_k(g) &= \int_{0}^{g} u^k \left(1+\frac{u^2}{2aA_1}\right)^{-\nu} du = \frac{1}{2}\int_{0}^{g^2} w^{(k+1)/2-1} \left(1+\frac{w}{2aA_1}\right)^{-\nu} dw. & (91)
\end{aligned}
$$

Using equations 3.194.1–3.194.3 from Gradshteyn and Ryzhik (2014) to this leads to

$$I_k(g) = \begin{cases} \frac{g^{k+1-2\nu}\,(2aA_1)^{\nu}}{2\nu-k-1}\, {}_2F_1\left(\nu, (2\nu-k-1)/2; (2\nu-k+1)/2; -2aA_1/g^2\right) & \text{if } g > 0, \\ (2aA_1)^{(k+1)/2}\,\mathrm{B}\left((k+1)/2, (2\nu-k-1)/2\right) & \text{if } g = 0, \end{cases} \tag{92}$$

and

$$J_k\,(g) = \frac{g^{k+1}}{k+1}\, {}_2F_1\left(\nu, (k+1)/2; (k+3)/2; -g^2/2aA_1\right) \quad \text{for } g > 0, \tag{93}$$

where ${}_2F_1$ is the Gauss hypergeometric function. To our interest, this results in

$$I_1(an\,|\bar{x}|) = \begin{cases} \frac{(an|\bar{x}|)^{2(1-\nu)}(2aA_1)^{\nu}}{2(\nu-1)}\, {}_2F_1\left(\nu, \nu-1; \nu; -2A_1/an^2\bar{x}^2\right) & \text{if } \bar{x} \neq 0, \\ 2aA_1\,\mathrm{B}\left(1, \nu-1\right) & \text{if } \bar{x} = 0. \end{cases} \tag{94}$$

This leads to

$$m_{11} = \begin{cases} 2an\bar{x}\, I_1\left(an\,|\bar{x}|\right) & \text{if } \bar{x} \neq 0, \\ 0 & \text{if } \bar{x} = 0 \end{cases} = \begin{cases} (2aA_1)^{3/2}\ f_2\, d\,|d|^{2(1-\nu)} & \text{if } \bar{x} \neq 0, \\ 0 & \text{if } \bar{x} = 0. \end{cases} \tag{95}$$

where $f_2 = {}_2F_1\left(\nu, \nu-1; \nu; -1/d^2\right) / \left(\nu - 1\right)$. Similarly, it also results in

$$I_2(an\,|\bar{x}|) = \begin{cases} (2aA_1)^{3/2}\ f_3\ |d|^{3-2\nu} & \text{if } \bar{x} \neq 0, \\ (2aA_1)^{3/2}\ \mathrm{B}\left(3/2, \nu-3/2\right) & \text{if } \bar{x} = 0, \end{cases} \tag{96}$$

and

$$J_2(an\,|\bar{x}|) = (2aA_1)^{3/2}\ f_4\ |d|^3\,, \tag{97}$$

where $f_3 = {}_2F_1\left(\nu, \nu-3/2; \nu-1/2; -1/d^2\right) / \left(2\nu-3\right)$ and $f_4 = {}_2F_1\left(\nu, 3/2; 5/2; -d^2\right)/3$.

This leads to

$$
m_{12} = \begin{cases} (2aA_1)^{3/2} \ f_3 \ |d|^{3-2\nu} & \text{if } \bar{x} < 0, \\ (2aA_1)^{3/2} \ \mathrm{B}\left(3/2, \nu - 3/2\right) & \text{if } \bar{x} = 0, \\ (2aA_1)^{3/2} \ \left(f_3 \ |d|^{3-2\nu} + 2f_4 \, |d|^3\right) & \text{if } \bar{x} > 0. \end{cases} \tag{98}
$$

Finally, (71) follows by combining m_{10}, m_{11} and m_{12}.

■

Theorem S2.10. *Under the assumptions in Section S2.2.2, the Bayes factor $BF_{10}(\mathbf{x})$ in favor of the alternative hypothesis H_1 against the null hypothesis H_0 is given by*

$$
\begin{cases} C_1 \left(f_1 d^2 \left(1 - F_{2\nu-1}\left(-d\sqrt{2\nu-1}\right)\right) + f_2 \, d \, |d|^{2(1-\nu)} + f_3 \ |d|^{3-2\nu}\right) & \text{if } \bar{x} < 0, \\ C_1 B\left(3/2, \nu - 3/2\right) & \text{if } \bar{x} = 0, \\ C_1 \left(f_1 d^2 \left(1 - F_{2\nu-1}\left(-d\sqrt{2\nu-1}\right)\right) + f_2 \, d \, |d|^{2(1-\nu)} + \right. \\ \quad \left. f_3 \ |d|^{3-2\nu} + 2f_4 \, |d|^3\right) & \text{if } \bar{x} > 0, \end{cases} \tag{99}
$$

where

$$
C_1 = \frac{4\Gamma\left(\nu\right)}{\sqrt{\pi}(n\tau^2 + 1)^{3/2}\Gamma\left(n/2 + \alpha\right)}, \tag{100}
$$

$\bar{x}$, S are as in (17), $\nu = (n+3)/2 + \alpha$, T, r, G and H are as in (63)–(64), $d = \sqrt{r}T/\sqrt{(n-1)H}$, and f_1 to f_4 are as in (73)–(74) with d is it is defined here.

Proof: Following the definition of the Bayes factor we know that $\mathrm{BF}_{10}(\mathbf{x}) = m_1(\mathbf{x})/m_0(\mathbf{x})$. While substituting the expression for the marginal density of $\mathbf{x}$ from Thm S2.6 and S2.9 we note that

$$\frac{c^*}{m_0(\mathbf{x})} = \frac{4\Gamma(\nu)}{\sqrt{\pi}(n\tau^2+1)^{3/2}\Gamma(n/2+\alpha)}\left(\frac{\beta+S/2+n\bar{x}^2/2}{\beta+S/2+n\bar{x}^2/2(n\tau^2+1)}\right)^{n/2+\alpha} \quad (101)$$

$$= \frac{4\Gamma(\nu)}{\sqrt{\pi}(n\tau^2+1)^{3/2}\Gamma(n/2+\alpha)}\left(\frac{G}{H}\right)^{n/2+\alpha}. \quad (102)$$

Also, d as in Theorem S2.9 can be rewritten as

$$d = \frac{\sqrt{r}n\bar{x}}{\sqrt{2(\beta+S/2+n\bar{x}^2/2(n\tau^2+1))}} = \frac{\sqrt{r}T}{\sqrt{(n-1)H}}. \quad (103)$$

(99) directly follows from combining these.

■

S3 Proofs of theorems of two-sample tests

S3.1 Variance known

S3.1.1 Two-sided tests

Suppose $\mathbf{x_1} = (x_{1,1}, \ldots, x_{1,n_1})$ and $\mathbf{x_2} = (x_{2,1}, \ldots, x_{1,n_2})$ are observations from i.i.d. $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ distributions, respectively, and we wish to test $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$. To this end, we assume that under both H_0 and H_1, the prior on μ_1 is $U(-a, a)$ for some large a. Under H_1, we further assume that $\mu_2 = \mu_1 + \delta$, where

$$p(\delta \mid \tau^2, \sigma^2) = \frac{1}{\sqrt{2\pi}\tau^3\sigma^3}\delta^2 \exp\left(-\frac{\delta^2}{2\tau^2\sigma^2}\right), \quad (104)$$

a normal moment prior on the difference between the means μ_1 and μ_2. We let $\phi(\cdot \mid \mu, \sigma^2)$ denote a normal density function with mean μ and variance σ^2. With a uniform prior on

μ_1 and sufficiently large a, we note that the marginal distributions described below are invariant with respect to the labeling of samples.

Theorem S3.1. *Under the assumptions above and assuming H_1 is true and that σ^2 is known, the marginal density of the data $m_1(\mathbf{x}_1, \mathbf{x_2} \mid \sigma^2)$ is given by*

$$\frac{\sqrt{2\pi}c_1}{\sigma\sqrt{n(m\tau^2+1)^3}}\left(\sigma^2 + \frac{m^2\tau^2(\bar{x_1}-\bar{x_2})^2}{m\tau^2+1}\right)\exp\left\{-\frac{1}{2\sigma^2}\left[\frac{m(\bar{x_1}-\bar{x_2})^2}{m\tau^2+1}\right]\right\}, \tag{105}$$

where we define the following quantities for $i = 1, 2$:

$$\bar{x_i} = \sum_{j=1}^{n_i} x_{j,i}/n_i, \qquad S_i = \sum_{j=1}^{n_i}(x_{j,i}-\bar{x_i})^2, \qquad n = n_1 + n_2 \tag{106}$$

$$m = \frac{n_1 n_2}{n_1+n_2}, \qquad c_1 = \frac{1}{2a}(2\pi\sigma^2)^{-(n_1+n_2)/2}\exp\left[-\frac{1}{2\sigma^2}\left(S_1+S_2\right)\right]. \tag{107}$$

Proof: When not indicated otherwise, we assume that all sums and products extend from $i = 1$ to 2, and that integrals extend from $-\infty$ to ∞. We also define

$$c_2(\delta) = c_1\, p(\delta \mid \tau^2, \sigma^2).$$

The marginal density $m_1(\mathbf{x}_1, \mathbf{x_2} \mid \sigma^2)$ (ignoring dependence on σ^2 and τ^2) can be expressed as

$$m_1(\mathbf{x}_1, \mathbf{x_2} \mid \sigma^2) \tag{108}$$

$$= \int\int_{-a}^{a}\frac{p(\delta \mid \tau^2,\sigma^2)}{2a}\prod_{j=1}^{n_1}\phi(x_{1,j} \mid \mu_1, \sigma^2)\prod_{j=1}^{n_2}\phi(x_{2,j} \mid \mu_1+\delta, \sigma^2)d\mu_1\, d\delta \tag{109}$$

$$\doteq \int\int c_2(\delta)\exp\left\{-\frac{1}{2\sigma^2}\left[n_1(\bar{x_1}-\mu_1)^2 + n_2(\bar{x_2}-\mu_1-\delta)^2\right]\right\} d\mu_1\, d\delta. \tag{110}$$

Defining $b = [n_1\bar{x_1} + n_2(\bar{x_2} - \delta)]/n$, completing the square in μ_1 and integrating leads to

$$= \int\int c_2(\delta)\exp\left\{-\frac{1}{2\sigma^2}\left[n(\mu_1 - b)^2 - nb^2 + n_1\bar{x_1}^2 + n_2(\bar{x_2} - \delta)^2\right]\right\} d\mu_1\, d\delta \quad (111)$$

$$= \int \frac{c_1\delta^2}{\tau^3\sigma^2\sqrt{n}}\exp\left\{-\frac{1}{2\sigma^2}\left[\frac{\delta^2}{\tau^2} + n_1\bar{x_1}^2 + n_2(\bar{x_2} - \delta)^2 - nb^2\right]\right\} d\delta. \quad (112)$$

Completing the square in δ and defining $d = [m(\bar{x_2} - \bar{x_1})]$ and $f = (m + 1/\tau^2)$ leads to

$$= \int \frac{c_1\delta^2}{\tau^3\sigma^2\sqrt{n}}\exp\left\{-\frac{1}{2\sigma^2}\left[m(\bar{x_1} - \bar{x_2})^2 - \frac{d^2}{f} + f\left(\delta - \frac{d}{f}\right)^2\right]\right\} d\delta \quad (113)$$

Noting that the integral is proportional to the second moment of a normal density with mean d/f and variance σ^2/f results in

$$= \frac{\sqrt{2\pi}c_1}{\tau^3\sigma\sqrt{nf}}\left(\frac{\sigma^2}{f} + \frac{d^2}{f^2}\right)\exp\left\{-\frac{1}{2\sigma^2}\left[\frac{m(\bar{x_1} - \bar{x_2})^2}{m\tau^2 + 1}\right]\right\}. \quad (114)$$

$$= \frac{\sqrt{2\pi}c_1}{\sigma\sqrt{n(m\tau^2 + 1)^3}}\left(\sigma^2 + \frac{m^2\tau^2(\bar{x_1} - \bar{x_2})^2}{m\tau^2 + 1}\right)\exp\left\{-\frac{1}{2\sigma^2}\left[\frac{m(\bar{x_1} - \bar{x_2})^2}{m\tau^2 + 1}\right]\right\} \quad (115)$$

■

Theorem S3.2. *Under the assumptions of Theorem S3.1, but now assuming H_0 to be true, the marginal density of the data is given by*

$$m_0(\mathbf{x}_1, \mathbf{x}_2 \mid \sigma^2) = \frac{\sqrt{2\pi}\sigma c_1}{\sqrt{n}}\exp\left\{-\frac{1}{2\sigma^2}\left[m(\bar{x_1} - \bar{x_2})^2\right]\right\}. \quad (116)$$

Proof: Using the proof of Theorem 1, divide equation (112) by $p(\delta \mid \tau^2, \sigma^2)$ and set $\delta = 0$ to obtain the marginal density of the data under H_0 after marginalizing over $\mu \sim U(-a, a)$. Simplifying the result in the exponential term yields (116).

■

Theorem S3.3. *Under the assumptions of Thm S3.1 and S3.2, the Bayes factor in favor of the alternative hypothesis H_1 against the null hypothesis H_0 is given by*

$$BF_{10}(\mathbf{x}_1, \mathbf{x}_2 \,|\, \sigma^2) = (m\tau^2 + 1)^{-3/2} \left(1 + rT^2\right) \exp\left(\frac{rT^2}{2}\right), \tag{117}$$

where $r = 1/\left(1 + (m\tau^2)^{-1}\right)$ and $T = \sqrt{m}(\bar{x}_2 - \bar{x}_1)/\sigma$.

Proof: Following the definition of the Bayes factor and substituting the expression for the marginal density of $(\mathbf{x}_1, \mathbf{x}_2)$ from Thm S3.1 and S3.2 leads to

$$\begin{aligned}
& \mathrm{BF}_{10}(\mathbf{x}_1, \mathbf{x}_2 \,|\, \sigma^2) && (118)\\
= \; & \frac{m_1(\mathbf{x}_1, \mathbf{x}_2 \,|\, \sigma^2)}{m_0(\mathbf{x}_1, \mathbf{x}_2 \,|\, \sigma^2)} && (119)\\
= \; & \frac{1}{\sigma^2(m\tau^2+1)^{3/2}} \left[\sigma^2 + \frac{m^2\tau^2(\bar{x}_1 - \bar{x}_2)^2}{m\tau^2+1}\right] \exp\left[\frac{m^2\tau^2(\bar{x}_1 - \bar{x}_2)^2}{2\sigma^2(m\tau^2+1)}\right] && (120)\\
= \; & (m\tau^2 + 1)^{-3/2} \left(1 + rT^2\right) \exp\left(\frac{rT^2}{2}\right). && (121)
\end{aligned}$$

■

S3.1.2 One-sided tests

Assume the same setup as in Section S3.1.1, except that we now wish to test $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_2 > \mu_1$. To this end, under both H_0 and H_1 we similarly assume the $U(-a, a)$ prior on μ_1 is for some large a. Under H_1 we still assume that $\mu_2 = \mu_1 + \delta$ except the prior on δ is assumed to be a normal moment prior truncated on $(0, \infty)$. The density is specified by

$$p_+(\delta \,|\, \tau^2, \sigma^2) = \frac{\sqrt{2}}{\sqrt{\pi}\tau^3\sigma^3}\delta^2 \exp\left(-\frac{\delta^2}{2\tau^2\sigma^2}\right) \quad \text{for } \delta > 0. \tag{122}$$

Under this setup we note that the marginal density of $\mathbf{x}$ under the null hypothesis $H_0 : \mu = 0$ is the same as in Theorem S3.2.

Theorem S3.4. *Under the assumptions above and assuming H_1 is true and that σ^2 is known, the marginal density of the data $m_1(\mathbf{x}_1, \mathbf{x_2} \,|\, \sigma^2)$ is given by*

$$\frac{\sqrt{2\pi}\sigma c_1}{\sqrt{n}\,(m\tau^2+1)^{3/2}} \exp\left(-\frac{d_1^2}{m\tau^2}\right)\left[\left(2d_1^2+1\right)(1-erf(-d_1)) + \frac{2d_1}{\sqrt{\pi}}\exp\left(-d_1^2\right)\right], \tag{123}$$

where m, $\bar{x_1}$, $\bar{x_2}$, S_1, S_2, c_1 are as in (106)–(107), $d = m(\bar{x_2} - \bar{x_1})$, $f = (m + 1/\tau^2)$, and $d_1 = d/\sigma\sqrt{2f}$.

Proof: The marginal density $m_1(\mathbf{x}_1, \mathbf{x_2} \,|\, \sigma^2)$ (ignoring dependence on σ^2 and τ^2) can be expressed as

$$m_1(\mathbf{x}_1, \mathbf{x_2} \,|\, \sigma^2) = \int_0^\infty \int_{-a}^{a} \frac{p_+(\delta \,|\, \tau^2, \sigma^2)}{2a} \prod_{j=1}^{n_1} \phi(x_{1,j} \,|\, \mu_1, \sigma^2) \prod_{j=1}^{n_2} \phi(x_{2,j} \,|\, \mu_1 + \delta, \sigma^2) d\mu_1\, d\delta \tag{124}$$

Marginalizing over μ_1 and following (112) leads to

$$m_1(\mathbf{x}_1, \mathbf{x_2} \,|\, \sigma^2) \doteq \int_0^\infty \frac{2c_1\delta^2}{\tau^3\sigma^2\sqrt{n}} \exp\left\{-\frac{1}{2\sigma^2}\left[\frac{\delta^2}{\tau^2} + n_1\bar{x_1}^2 + n_2(\bar{x_2}-\delta)^2 - nb^2\right]\right\} d\delta, \tag{125}$$

where c_1 is as in (107). Completing the square in δ and defining $d = [m(\bar{x_2} - \bar{x_1})]$ and $f = (m + 1/\tau^2)$ lead to

$$\begin{aligned} & m_1(\mathbf{x}_1, \mathbf{x_2} \,|\, \sigma^2) & (126)\\ = & \int_0^\infty \frac{2c_1\delta^2}{\tau^3\sigma^2\sqrt{n}} \exp\left\{-\frac{1}{2\sigma^2}\left[m(\bar{x_1}-\bar{x_2})^2 - \frac{d^2}{f} + f\left(\delta - \frac{d}{f}\right)^2\right]\right\} d\delta & (127)\\ = & \frac{2c_1}{\tau^3\sigma^2\sqrt{n}} \exp\left\{-\frac{1}{2\sigma^2}\left[\frac{m(\bar{x_1}-\bar{x_2})^2}{m\tau^2+1}\right]\right\} \int_0^\infty \delta^2 \exp\left[-\frac{f\,(\delta - d/f)^2}{2\sigma^2}\right] d\delta. & (128) \end{aligned}$$

Using 2.1.3.1 from Korotkov and Korotkov (2020) and some algebraic simplifications result in

$$
\begin{aligned}
& m_1(\mathbf{x}_1, \mathbf{x_2} \,|\, \sigma^2) && (129)\\
= \; & \frac{2c_1}{\tau^3\sigma^2\sqrt{n}} \exp\left\{-\frac{1}{2\sigma^2}\left[\frac{m(\bar{x_1}-\bar{x_2})^2}{m\tau^2+1}\right]\right\} \times && (130)\\
& \left[\frac{\sqrt{\pi}\sigma^3}{\sqrt{2}f^{3/2}}\left(\frac{d^2}{\sigma^2 f}+1\right)\left\{1-\mathrm{erf}\left(-\frac{d}{\sigma\sqrt{2f}}\right)\right\}+\frac{d\sigma^2}{f^2}\exp\left(-\frac{d^2}{2\sigma^2 f}\right)\right] && (131)\\
= \; & \frac{\sqrt{2\pi}\sigma c_1}{\sqrt{n}\,(m\tau^2+1)^{3/2}} \exp\left(-\frac{d_1^2}{m\tau^2}\right)\left[\left(2d_1^2+1\right)(1-\mathrm{erf}\,(-d_1))+\frac{2d_1}{\sqrt{\pi}}\exp\left(-d_1^2\right)\right] && (132)
\end{aligned}
$$

∎

Theorem S3.5. *Under the assumptions in Section S3.1.2, the Bayes factor* $BF_{10}(\mathbf{x}_1, \mathbf{x}_2 \,|\, \sigma^2)$ *in favor of the alternative hypothesis* H_1 *against the null hypothesis* H_0 *is given by*

$$
\left(m\tau^2+1\right)^{-3/2}\exp\left(\frac{rT^2}{2}\right)\left[\left(rT^2+1\right)\left(1-erf\left(-\frac{\sqrt{r}T}{\sqrt{2}}\right)\right)+\frac{\sqrt{2r}T}{\sqrt{\pi}}\exp\left(-\frac{rT^2}{2}\right)\right], \tag{133}
$$

where $r = 1/\left(1+(m\tau^2)^{-1}\right)$ *and* $T = \sqrt{m}(\bar{x_2}-\bar{x_1})/\sigma$.

Proof: Following the definition of the Bayes factor and substituting the expression for the marginal density of $(\mathbf{x}_1, \mathbf{x}_2)$ from Thm S3.4 and S3.2 leads to

$$
\begin{aligned}
& \mathrm{BF}_{10}(\mathbf{x}_1, \mathbf{x}_2 \,|\, \sigma^2) && (134)\\
= & \frac{m_1(\mathbf{x}_1, \mathbf{x}_2 \,|\, \sigma^2)}{m_0(\mathbf{x}_1, \mathbf{x}_2 \,|\, \sigma^2)} && (135)\\
= & \left(m\tau^2 + 1\right)^{-3/2} \exp\left(\frac{m^2\tau^2(\bar{x_2} - \bar{x_1})^2}{2\sigma^2(m\tau^2+1)}\right) \times && (136)\\
& \left[\left(\frac{m^2\tau^2(\bar{x_2} - \bar{x_1})^2}{\sigma^2(m\tau^2+1)} + 1\right)\left(1 - \operatorname{erf}\left(-\frac{m\tau(\bar{x_2} - \bar{x_1})}{\sigma\sqrt{2(m\tau^2+1)}}\right)\right) + \right. && (137)\\
& \left. \frac{\sqrt{2}m\tau(\bar{x_2} - \bar{x_1})}{\sigma\sqrt{\pi(m\tau^2+1)}} \exp\left(-\frac{m^2\tau^2(\bar{x_2} - \bar{x_1})^2}{2\sigma^2(m\tau^2+1)}\right)\right] && (138)\\
= & \left(m\tau^2 + 1\right)^{-3/2} \exp\left(\frac{rT^2}{2}\right) \times && (139)\\
& \left[\left(rT^2 + 1\right)\left(1 - \operatorname{erf}\left(-\frac{\sqrt{r}T}{\sqrt{2}}\right)\right) + \frac{\sqrt{2r}T}{\sqrt{\pi}} \exp\left(-\frac{rT^2}{2}\right)\right]. && (140)
\end{aligned}
$$

■

S3.2 Variance unknown

S3.2.1 Two-sided tests

We now consider the case where the variance σ^2 is not known. In this case, we assume that σ^2 is drawn *a priori* from an inverse gamma density parameterized as in (51).

Theorem S3.6. *Under the assumptions stated above and assuming H_1 is true, the marginal density of the data is given by*

$$
m_1(\mathbf{x}_1, \mathbf{x}_2) = \frac{c_3 c_4^{-(n+1)/2-\alpha}}{(m\tau^2+1)^{3/2}} \left[c_4 + \frac{m^2\tau^2(\bar{x}_1 - \bar{x}_2)^2}{m\tau^2+1}\left(\frac{n-1}{2} + \alpha\right)\right], \tag{141}
$$

where $\bar{x}_1$, $\bar{x}_2$, S_1, S_2, n, m *are defined in (106-107), and*

$$c_3 = \frac{(2\pi)^{-(n-1)/2}\,\beta^\alpha}{2a\sqrt{n}\,\Gamma(\alpha)}\,\Gamma\Big(\frac{n-1}{2}+\alpha\Big), \quad \textit{and} \;\; c_4 = \frac{m(\bar{x}_1-\bar{x}_2)^2}{2(m\tau^2+1)} + \frac{S_1+S_2}{2} + \beta. \tag{142}$$

Proof: The marginal density $m_1(\mathbf{x}_1, \mathbf{x}_2)$ (ignoring dependence on τ^2) can be expressed as

$$m_1(\mathbf{x}_1, \mathbf{x}_2) = \int\int\int \pi(\sigma^2 \,|\, \alpha, \beta)\, \pi(\mu_1 \,|\, a)\, \pi(\delta \,|\, \tau^2, \sigma^2) \times \tag{143}$$

$$\prod_{j=1}^{n_1} \phi(x_{1,j} \,|\, \mu_1, \sigma^2) \prod_{j=1}^{n_2} \phi(x_{2,j} \,|\, \mu_1+\delta, \sigma^2)\, d\mu_1\, d\delta\, d\sigma^2. \tag{144}$$

To this we note that, given σ^2 the integral with respect to μ_1 and δ is identical to (109). From Theorem S3.1, and noting that the integrals with respect to σ^2 are proportional to an inverse gamma density yields

$$\begin{aligned} m_1(\mathbf{x}_1, \mathbf{x}_2) = \;& \frac{(2\pi)^{-(n_1+n_2-1)/2}\beta^\alpha}{2a\sqrt{n(m\tau^2+1)^3}\,\Gamma(\alpha)} \int (\sigma^2)^{-(n_1+n_2+1)/2-\alpha-1} \left(\sigma^2 + \frac{m^2\tau^2(\bar{x}_1-\bar{x}_2)^2}{m\tau^2+1}\right) \\ & \times \exp\left[-\frac{1}{\sigma^2}\left\{\frac{m(\bar{x}_1-\bar{x}_2)^2}{2(m\tau^2+1)} + \frac{S_1+S_2}{2} + \beta\right\}\right] d\sigma^2 \qquad (145) \\ = \;& \frac{(2\pi)^{-(n_1+n_2-1)/2}\beta^\alpha}{2a\sqrt{n(m\tau^2+1)^3}\,\Gamma(\alpha)} \left[\Gamma\Big(\frac{n_1+n_2+1}{2}+\alpha-1\Big) c_4^{-(n_1+n_2+1)/2-\alpha+1}\right. \\ & + \left.\frac{m^2\tau^2(\bar{x}_1-\bar{x}_2)^2}{m\tau^2+1}\,\Gamma\Big(\frac{n_1+n_2+1}{2}+\alpha\Big) c_4^{-(n_1+n_2+1)/2-\alpha}\right] \qquad (146) \\ = \;& \frac{c_3 c_4^{-(n+1)/2-\alpha}}{\sqrt{n}} \left[c_4 + \frac{m^2\tau^2(\bar{x}_1-\bar{x}_2)^2}{m\tau^2+1}\left(\frac{n-1}{2}+\alpha\right)\right]. \qquad (147) \end{aligned}$$

■

Theorem S3.7. *Under the assumptions of Theorem S3.6, but now assuming* H_0 *to be true, the marginal density of the data is given by*

$$m_0(\mathbf{x}_1, \mathbf{x}_2) = c_3 \left[\frac{m(\bar{x}_1 - \bar{x}_2)^2}{2} + \frac{S_1 + S_2}{2} + \beta\right]^{-(n-1)/2-\alpha}, \tag{148}$$

where c_3 is as in (142).

Proof: The marginal density $m_0(\mathbf{x}_1, \mathbf{x}_2)$ can be expressed as

$$m_0(\mathbf{x}_1, \mathbf{x}_2) = \int\int \pi(\sigma^2 \,|\, \alpha, \beta)\, \pi(\mu \,|\, a) \prod_{i=1}^{2} \prod_{j=1}^{n_i} \phi(x_{i,j} \,|\, \mu, \sigma^2)\, d\mu\, d\sigma^2. \tag{149}$$

To this we note that, given σ^2 the integral with respect to μ is the same as the marginal $m_0(\mathbf{x}_1, \mathbf{x}_2 \,|\, \sigma^2)$ in Theorem (S3.2). Using (116) and noting that the integrals with respect to σ^2 are proportional to an Inverse-gamma density results in

$$\begin{aligned} m_0(\mathbf{x}_1, \mathbf{x}_2) = & \frac{(2\pi)^{-(n_1+n_2-1)/2}\beta^\alpha}{2a\sqrt{n}\,\Gamma(\alpha)} \int (\sigma^2)^{-(n_1+n_2-1)/2-\alpha-1} \times & (150) \\ & \exp\left[-\frac{1}{\sigma^2}\left\{\frac{m(\bar{x}_1 - \bar{x}_2)^2}{2} + \frac{S_1 + S_2}{2} + \beta\right\}\right] d\sigma^2 & (151) \\ = & \frac{(2\pi)^{-(n_1+n_2-1)/2}\beta^\alpha}{2a\sqrt{n}\,\Gamma(\alpha)} \Gamma\left(\frac{n_1 + n_2 - 1}{2} + \alpha\right) \times & (152) \\ & \left[\frac{m(\bar{x}_1 - \bar{x}_2)^2}{2} + \frac{S_1 + S_2}{2} + \beta\right]^{-(n_1+n_2-1)/2-\alpha} & (153) \\ = & c_3 \left[\frac{m(\bar{x}_1 - \bar{x}_2)^2}{2} + \frac{S_1 + S_2}{2} + \beta\right]^{-(n-1)/2-\alpha}. & (154) \end{aligned}$$

■

Theorem S3.8. *Under the assumptions of Thm S3.6 and S3.7, the Bayes factor in favor of the alternative hypothesis H_1 against the null hypothesis H_0 is given by*

$$BF_{10}(\mathbf{x}_1, \mathbf{x}_2) = (m\tau^2 + 1)^{-3/2} \left(\frac{G_2}{H_2}\right)^{(n-1)/2+\alpha} \left(1 + \frac{qT_2^2}{H_2}\right), \tag{155}$$

where $\bar{x}_1$, $\bar{x}_2$, S_1, S_2, n, m *are defined in (106)–(107), and*

$$r = \frac{m\tau^2}{m\tau^2+1}, \qquad q = \frac{2r\left((n-1)/2+\alpha\right)}{n-2}, \qquad S = S_1 + S_2, \tag{156}$$

$$T = \frac{\sqrt{m}(\bar{x}_1-\bar{x}_2)}{\sqrt{S/(n-2)}}, \qquad G = 1 + \frac{T^2}{n-2} + \frac{2\beta}{S}, \qquad H = 1 + \frac{(1-r)T^2}{n-2} + \frac{2\beta}{S}. \tag{157}$$

Proof: Following the definition of the Bayes factor and substituting the expression for the marginal density of $(\mathbf{x}_1, \mathbf{x}_2)$ from Thm S3.6 and S3.7 leads to

$$\begin{aligned}
& \mathrm{BF}_{10}(\mathbf{x}_1,\mathbf{x}_2) && (158)\\
= \;& \frac{m_1(\mathbf{x}_1,\mathbf{x}_2)}{m_0(\mathbf{x}_1,\mathbf{x}_2)} && (159)\\
= \;& \frac{1}{(m\tau^2+1)^{3/2}}\left[\frac{m(\bar{x}_1-\bar{x}_2)^2/2+S/2+\beta}{m(\bar{x}_1-\bar{x}_2)^2/2(m\tau^2+1)+S/2+\beta}\right]^{(n-1)/2+\alpha}\times && (160)\\
& \left[1+\frac{m^2\tau^2(\bar{x}_1-\bar{x}_2)^2\left((n-1)/2+\alpha\right)/(m\tau^2+1)}{m(\bar{x}_1-\bar{x}_2)^2/2(m\tau^2+1)+S/2+\beta}\right] && (161)\\
= \;& (m\tau^2+1)^{-3/2}\left[\frac{1+T^2/(n-2)+2\beta/S}{1+T^2/\left\{(n-2)(m\tau^2+1)\right\}+2\beta/S}\right]^{(n-1)/2+\alpha}\times && (162)\\
& \left[1+\frac{2m\tau^2\left((n-1)/2+\alpha\right)}{(m\tau^2+1)}\frac{T^2/(n-2)}{1+T^2/\left\{(n-2)(m\tau^2+1)\right\}+2\beta/S}\right] && (163)\\
= \;& (m\tau^2+1)^{-3/2}\left(\frac{G}{H}\right)^{(n-1)/2+\alpha}\left(1+\frac{qT^2}{H}\right). && (164)
\end{aligned}$$

■

S3.2.2 One-sided tests

Assume the same setup as in Section S3.2.1, except that we now wish to test $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_2 > \mu_1$. To this end, under both H_0 and H_1 we similarly assume the $U(-a, a)$ prior on μ_1 is for some large a. Under H_1 we still assume that $\mu_2 = \mu_1 + \delta$, but the prior on δ given σ^2 is assumed to be a normal moment prior truncated on $(0, \infty)$ whose density is defined by (122). To complete the model specification, under both H_0 and H_1 we again assume an inverse gamma prior on σ^2 defined by (51). Under these assumptions and assuming H_0 to be true, the marginal density of the data $m_0(\mathbf{x}_1, \mathbf{x}_2)$ is the same as in Theorem S3.7.

Theorem S3.9. *Under the assumptions stated above and assuming H_1 is true, the marginal density of the data $m_1(\mathbf{x}_1, \mathbf{x}_2)$ is given by*

$$\begin{cases} c^*\left(f_1 d_1^2\left(1 - F_{2\nu-1}\left(-d_1\sqrt{2\nu-1}\right)\right) + f_2\, d_1\, |d_1|^{2(1-\nu)} + f_3\; |d_1|^{3-2\nu}\right) & \text{if } \bar{x_2} < \bar{x_1}, \\ c^* B\left(3/2, \nu - 3/2\right) & \text{if } \bar{x_2} = \bar{x_1}, \\ c^*\left(f_1 d_1^2\left(1 - F_{2\nu-1}\left(-d_1\sqrt{2\nu-1}\right)\right) + f_2\, d_1\, |d_1|^{2(1-\nu)} + \right. & \\ \quad \left. f_3\; |d_1|^{3-2\nu} + 2 f_4\, |d_1|^3\right) & \text{if } \bar{x_2} > \bar{x_1}, \end{cases} \tag{165}$$

where $\bar{x}_1$, $\bar{x}_2$, S_1, S_2, n, m are defined in (106-107), $S = S_1 + S_2$, $d = m\left(\bar{x_2} - \bar{x_1}\right)$, $f = m + \tau^{-2}$, $d_1 = d/\sqrt{2fA_1}$, $\nu = n/2 + \alpha + 1$,

$$c^* = \frac{2^{3/2}(2\pi)^{-n/2}\,\beta^\alpha\,\Gamma(\nu)}{a\Gamma(\alpha)\sqrt{n}(m\tau^2+1)^{3/2}\,A_1^{(n-1)/2+\alpha}}, \quad A_1 = \beta + \frac{S}{2} + \frac{d^2}{2m\left(m\tau^2+1\right)}, \tag{166}$$

$$f_1 = B\left(\nu - 1/2, 1/2\right), \quad f_2 = \frac{{}_2F_1\left(\nu, \nu-1; \nu; -1/d_1^2\right)}{(\nu-1)}, \tag{167}$$

$$f_3 = \frac{{}_2F_1\left(\nu, \nu-3/2; \nu-1/2; -1/d_1^2\right)}{2\nu-3}, \quad f_4 = \frac{{}_2F_1\left(\nu, 3/2; 5/2; -d_1^2\right)}{3}, \tag{168}$$

$B(\cdot,\cdot)$ is the Beta function, $F_{2\nu-1}$ is the cdf of the Student's t distribution (center 0 and scale 1) with degrees of freedom $2\nu-1$, and ${}_2F_1$ is the Gauss hypergeometric function.

Proof: The marginal density $m_1(\mathbf{x}_1, \mathbf{x}_2)$ (ignoring dependence on τ^2) can be expressed as

$$m_1(\mathbf{x}_1, \mathbf{x}_2) = \int_0^\infty \int_0^\infty \int_{-a}^{a} p_+(\delta \,|\, \tau^2, \sigma^2)\, \pi(\sigma^2 \,|\, \alpha, \beta)\, \pi(\mu_1 \,|\, a) \times \tag{169}$$

$$\prod_{j=1}^{n_1} \phi(x_{1,j} \,|\, \mu_1, \sigma^2) \prod_{j=1}^{n_2} \phi(x_{2,j} \,|\, \mu_1 + \delta, \sigma^2)\, d\mu_1\, d\sigma^2\, d\delta. \tag{170}$$

Following (125) integrating with respect to μ_1, and then integrating with respect to σ^2 leads to

$$m_1(\mathbf{x}_1, \mathbf{x_2}) \doteq \int_0^\infty \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\alpha-1} \exp\left(-\frac{\beta}{\sigma^2}\right) \frac{(2\pi\sigma^2)^{-n/2}}{2a} \exp\left(-\frac{S}{2\sigma^2}\right) \times \tag{171}$$

$$\frac{2\delta^2}{\tau^3\sigma^2\sqrt{n}} \exp\left\{-\frac{1}{2\sigma^2}\left[\frac{\delta^2}{\tau^2} + n_1\bar{x_1}^2 + n_2(\bar{x_2} - \delta)^2 - nb^2\right]\right\} d\sigma^2\, d\delta \tag{172}$$

$$= \frac{(2\pi)^{-n/2}\beta^\alpha\Gamma(n/2 + \alpha + 1)}{a\Gamma(\alpha)\tau^3\sqrt{n}} \times \tag{173}$$

$$\int_0^\infty \delta^2 \left[\beta + \frac{1}{2}\left\{S + \frac{d^2}{m(m\tau^2 + 1)} + f\left(\delta - \frac{d}{f}\right)^2\right\}\right]^{-(n/2+\alpha+1)} d\delta. \tag{174}$$

Define $\nu = n/2 + \alpha + 1$, $A_1 = \beta + S/2 + d^2/2m(m\tau^2 + 1)$ and

$$A_2 = \frac{(2\pi)^{-n/2}\beta^\alpha\Gamma(\nu)}{a\Gamma(\alpha)\tau^3\sqrt{n}\, A_1^\nu}. \tag{175}$$

Then $m_1(\mathbf{x}_1, \mathbf{x_2})$ simplifies to

$$m_1(\mathbf{x}_1, \mathbf{x_2}) = A_2 \int_0^\infty \delta^2 \left(1 + \frac{f(\delta - d/f)^2}{2A_1}\right)^{-\nu} d\mu \tag{176}$$

$$= A_2 \int_{-d/f}^\infty \left(u + \frac{d}{f}\right)^2 \left(1 + \frac{fu^2}{2A_1}\right)^{-\nu} du \tag{177}$$

$$= A_2 \left(\frac{d^2}{f^2} I_0\left(-\frac{d}{f}\right) + \frac{2d}{f} I_1\left(-\frac{d}{f}\right) + I_2\left(-\frac{d}{f}\right)\right) \tag{178}$$

$$= A_2 \left(m_{10} + m_{11} + m_{12}\right), \tag{179}$$

where

$$I_k(g) = \int_g^\infty u^k \left(1 + \frac{fu^2}{2A_1}\right)^{-\nu} du \quad \text{for } g \in \mathbb{R}, k \geq 0, \tag{180}$$

$$m_{10} = \frac{d^2}{f^2} I_0\left(-\frac{d}{f}\right), \quad m_{11} = \frac{2d}{f} I_1\left(-\frac{d}{f}\right), \quad m_{12} = I_2\left(-\frac{d}{f}\right). \tag{181}$$

For $I_0(-d/f)$, first doing a change of variable with $w/\sqrt{2\nu - 1} = \sqrt{f}u/\sqrt{2A_1}$ and then some algebraic simplifications lead to

$$I_0(-d/f) = \int_{-d/f}^\infty \left(1 + \frac{fu^2}{2A_1}\right)^{-\nu} du \tag{182}$$

$$= \left(\frac{2A_1}{f(2\nu - 1)}\right)^{1/2} \int_{-d\sqrt{\frac{(2\nu-1)}{2fA_1}}}^\infty \left(1 + \frac{w^2}{2\nu - 1}\right)^{-((2\nu-1)+1)/2} dw \tag{183}$$

$$= \left(\frac{2A_1}{f}\right)^{1/2} \mathrm{B}\left((2\nu - 1)/2, 1/2\right) \left[1 - F_{2\nu-1}\left(-d\sqrt{\frac{(2\nu - 1)}{2fA_1}}\right)\right], \tag{184}$$

where $\mathrm{B}(\cdot, \cdot)$ is the Beta function and $F_{2\nu-1}$ is the cdf of the Student's t distribution (center 0 and scale 1) with degrees of freedom $2\nu - 1$. Following this we get

$$m_{10} = \frac{d^2}{f^2} I_0\left(-\frac{d}{f}\right) = \left(\frac{2A_1}{f}\right)^{3/2} f_1\, d_1^2 \left(1 - F_{2\nu-1}\left(-d_1\sqrt{2\nu - 1}\right)\right), \tag{185}$$

where $d_1 = d/\sqrt{2fA_1}$ and $f_1 = \mathrm{B}\left(\nu - 1/2, 1/2\right)$. To calculate $I_1(-d/f)$ and $I_2(-d/f)$ we again use (89)–(93). Using these we get

$$I_1(|d|/f) = \begin{cases} \frac{|d|^{2(1-\nu)}(2A_1)^\nu}{f^{2-\nu}\,2(\nu-1)}\, {}_2F_1\left(\nu, \nu-1; \nu; -2fA_1/d^2\right) & \text{if } \bar{x_2} \neq \bar{x_1}, \\ \frac{2A_1}{f}\,\mathrm{B}\left(1, \nu-1\right) & \text{if } \bar{x_2} = \bar{x_1}. \end{cases} \tag{186}$$

This leads to

$$m_{11} = \begin{cases} (2d/f)\ I_1\left(|d|/f\right) & \text{if } \bar{x_2} \neq \bar{x_1}, \\ 0 & \text{if } \bar{x_2} = \bar{x_1} \end{cases} = \begin{cases} \left(2A_1/f\right)^{3/2}\ f_2\, d_1 \left|d_1\right|^{2(1-\nu)} & \text{if } \bar{x_2} \neq \bar{x_1}, \\ 0 & \text{if } \bar{x_2} = \bar{x_1}. \end{cases} \tag{187}$$

where $f_2 = {}_2F_1\left(\nu, \nu-1; \nu; -1/d_1^2\right)/\left(\nu-1\right)$. Similarly, it also results in

$$I_2(|d|/f) = \begin{cases} \left(2A_1/f\right)^{3/2}\ f_3\ \left|d_1\right|^{3-2\nu} & \text{if } \bar{x_2} \neq \bar{x_1}, \\ \left(2A_1/f\right)^{3/2}\ \mathrm{B}\left(3/2, \nu-3/2\right) & \text{if } \bar{x_2} = \bar{x_1}, \end{cases} \tag{188}$$

and

$$J_2(|d|/f) = \left(2A_1/f\right)^{3/2}\ f_4\ \left|d_1\right|^3, \tag{189}$$

where $f_3 = {}_2F_1\left(\nu, \nu-3/2; \nu-1/2; -1/d_1^2\right)/\left(2\nu-3\right)$ and $f_4 = {}_2F_1\left(\nu, 3/2; 5/2; -d_1^2\right)/3$. This leads to

$$m_{12} = \begin{cases} \left(2A_1/f\right)^{3/2}\ f_3\ \left|d_1\right|^{3-2\nu} & \text{if } \bar{x_2} < \bar{x_1}, \\ \left(2A_1/f\right)^{3/2}\ \mathrm{B}\left(3/2, \nu-3/2\right) & \text{if } \bar{x_2} = \bar{x_1}, \\ \left(2A_1/f\right)^{3/2}\ \left(f_3\ \left|d_1\right|^{3-2\nu} + 2f_4\left|d_1\right|^3\right) & \text{if } \bar{x_2} > \bar{x_1}. \end{cases} \tag{190}$$

Finally, (71) follows by combining m_{10}, m_{11} and m_{12}.

■

Theorem S3.10. *Under the assumptions in Section S3.2.2, the Bayes factor* $BF_{10}(\mathbf{x}_1, \mathbf{x}_2)$ *in favor of the alternative hypothesis* H_1 *against the null hypothesis* H_0 *is given by*

$$\begin{cases} C_1 \left(f_1 d_1^2 \left(1 - F_{2\nu-1}\left(-d_1\sqrt{2\nu - 1}\right)\right) + f_2\, d_1 \left|d_1\right|^{2(1-\nu)} + f_3 \left|d_1\right|^{3-2\nu} \right) & \text{if } \bar{x_2} < \bar{x_1}, \\ C_1 B\left(3/2, \nu - 3/2\right) & \text{if } \bar{x_2} = \bar{x_1}, \\ C_1 \left(f_1 d_1^2 \left(1 - F_{2\nu-1}\left(-d_1\sqrt{2\nu - 1}\right)\right) + f_2\, d_1 \left|d_1\right|^{2(1-\nu)} + \right. & \\ \quad \left. f_3 \left|d_1\right|^{3-2\nu} + 2 f_4 \left|d_1\right|^{3} \right) & \text{if } \bar{x_2} > \bar{x_1}, \end{cases} \tag{191}$$

where

$$C_1 = \frac{2\Gamma\left(\nu\right)}{\sqrt{\pi}(m\tau^2 + 1)^{3/2}\Gamma\left((n-1)/2 + \alpha\right)}, \tag{192}$$

where $\bar{x}_1$, $\bar{x}_2$, S, n, m, ν *are as in Theorem S3.9,* T, r, G *and* H *are as in (156)–(157),* $d_1 = \sqrt{r}T/\sqrt{(n-2)H}$, *and* f_1 *to* f_4 *are as in (167)–(168) with* d_1 *is as it is defined here.*

Proof: Following the definition of the Bayes factor we know that $\mathrm{BF}_{10}(\mathbf{x}_1, \mathbf{x}_2) = m_1(\mathbf{x}_1, \mathbf{x}_2)/m_0(\mathbf{x}_1, \mathbf{x}_2)$. While substituting the expression for the marginal density of $(\mathbf{x}_1, \mathbf{x}_2)$ from Theorem S3.7 and S3.9 we note that

$$\begin{aligned} \frac{c^*}{m_0(\mathbf{x}_1, \mathbf{x}_2)} &= \frac{2\Gamma\left(\nu\right)}{\sqrt{\pi}(m\tau^2 + 1)^{3/2}\Gamma\left((n-1)/2 + \alpha\right)} \times && (193) \\ & \quad \left(\frac{\beta + S/2 + m\left(\bar{x_2} - \bar{x_1}\right)^2/2}{\beta + S/2 + m\left(\bar{x_2} - \bar{x_1}\right)^2/2(m\tau^2 + 1)} \right)^{(n-1)/2+\alpha} && (194) \\ &= \frac{2\Gamma\left(\nu\right)}{\sqrt{\pi}(m\tau^2 + 1)^{3/2}\Gamma\left((n-1)/2 + \alpha\right)} \left(\frac{G}{H}\right)^{(n-1)/2+\alpha}. && (195) \end{aligned}$$

Also, d_1 as in Theorem S3.9 can be rewritten as

$$d_1 = \frac{m\tau\left(\bar{x_2} - \bar{x_1}\right)}{\sqrt{2(m\tau^2+1)\left(\beta + S/2 + m\left(\bar{x_2} - \bar{x_1}\right)^2 / 2(m\tau^2+1)\right)}} = \frac{\sqrt{r}T}{\sqrt{(n-2)H}}. \quad (196)$$

(191) directly follows from combining these.

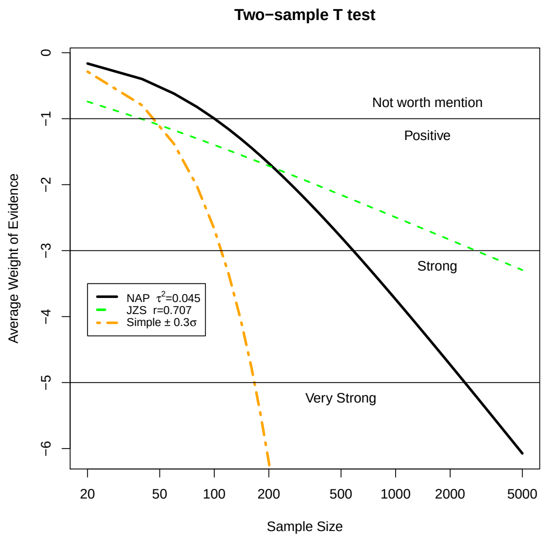
■

S4 Operating characteristics of one-sample z and two-sample t tests

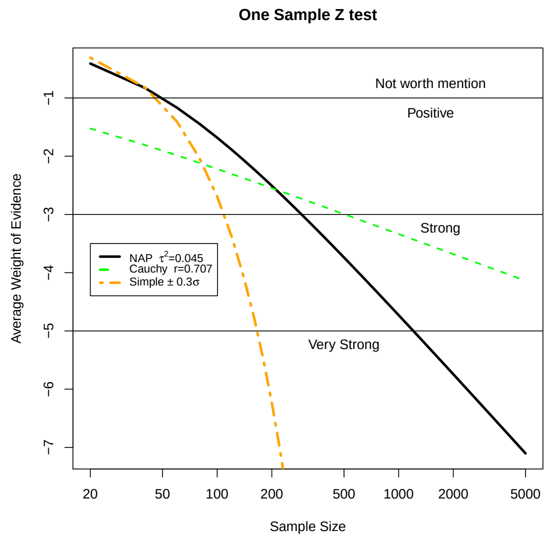
The operating characteristics for one-sample z, and two-sample z and t tests are similar to those cited in the main article for one-sample t tests. For purposes of comparison, plots similar to those found in the main article are presented below.

S4.1 Fixed design tests

Fig. S1 displays the operating characteristics of the one-sample z and two-sample t tests under a true null hypothesis. For the two-sample t test, equal sample sizes were assumed drawn from both populations, and the sample size appearing on the horizontal axis refers to the sample size for each sample. This figure is comparable to Fig. 2 in the main article for the default choices of the NAP and JZS priors.



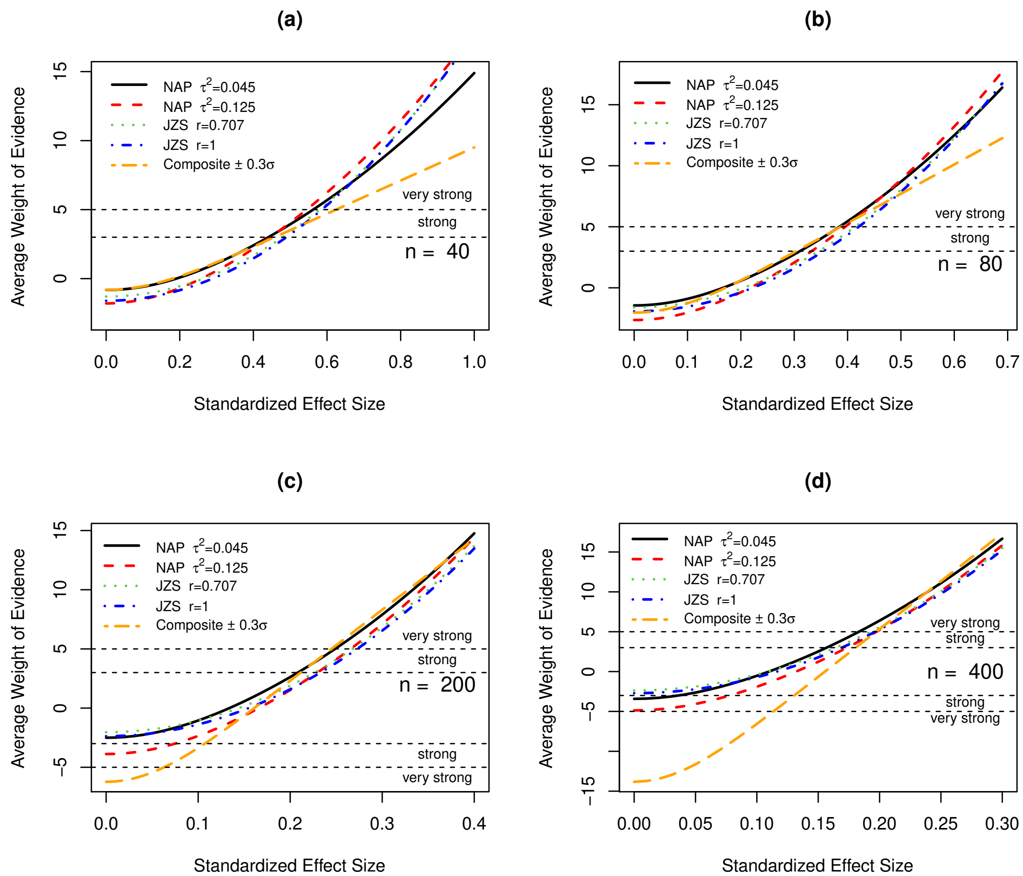
(a)



(b)

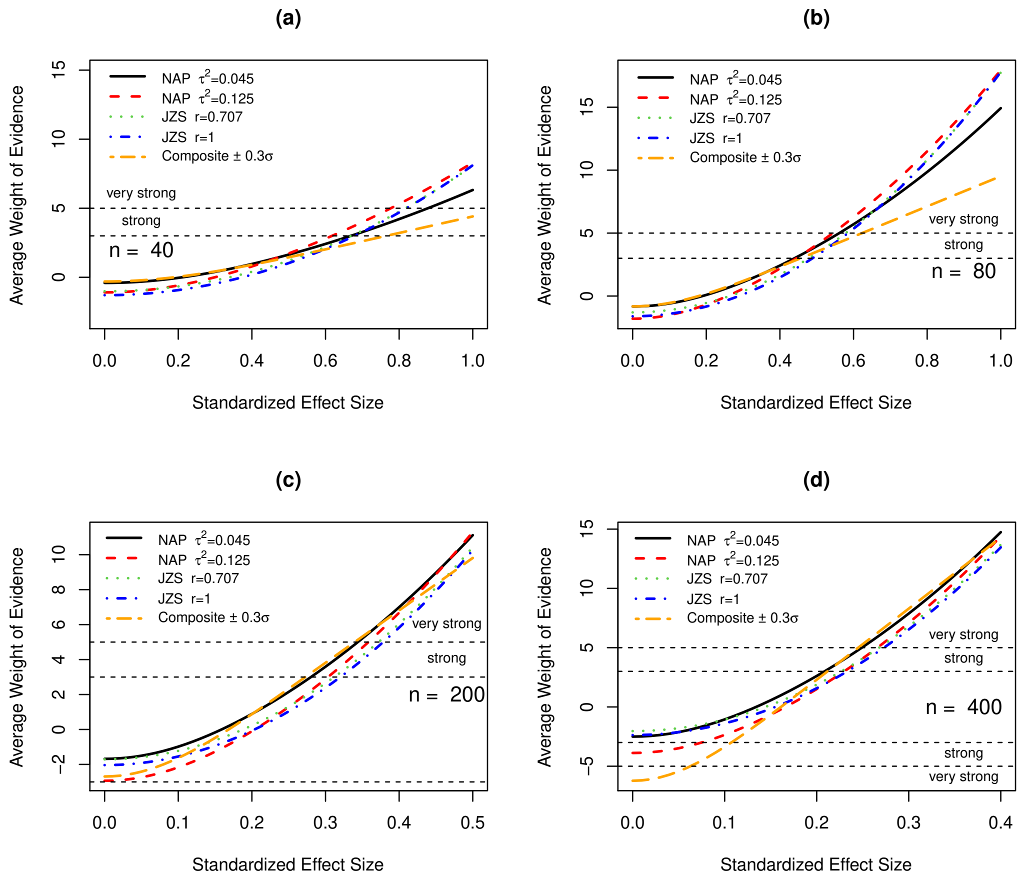
Figure S1: Weight of evidence for true null hypotheses in two-sample t test and one-sample z test. The black curves represent the average weight of evidence for the default NAP priors, while the dashed green curve the default JZS prior. The dashed orange curve depicts the average weight of evidence obtained when the alternative hypothesis assigned one-half mass to $\pm 0.3\sigma$.

For the same tests, Fig. S2–S4 displays the weight of evidence for different effect sizes under the alternative hypothesis as sample size varies. These figures are comparable to Fig. 3 in the main article for the composite alternative placing one-half mass at $\pm 0.3\sigma$ and different choices of the NAP and JZS priors.



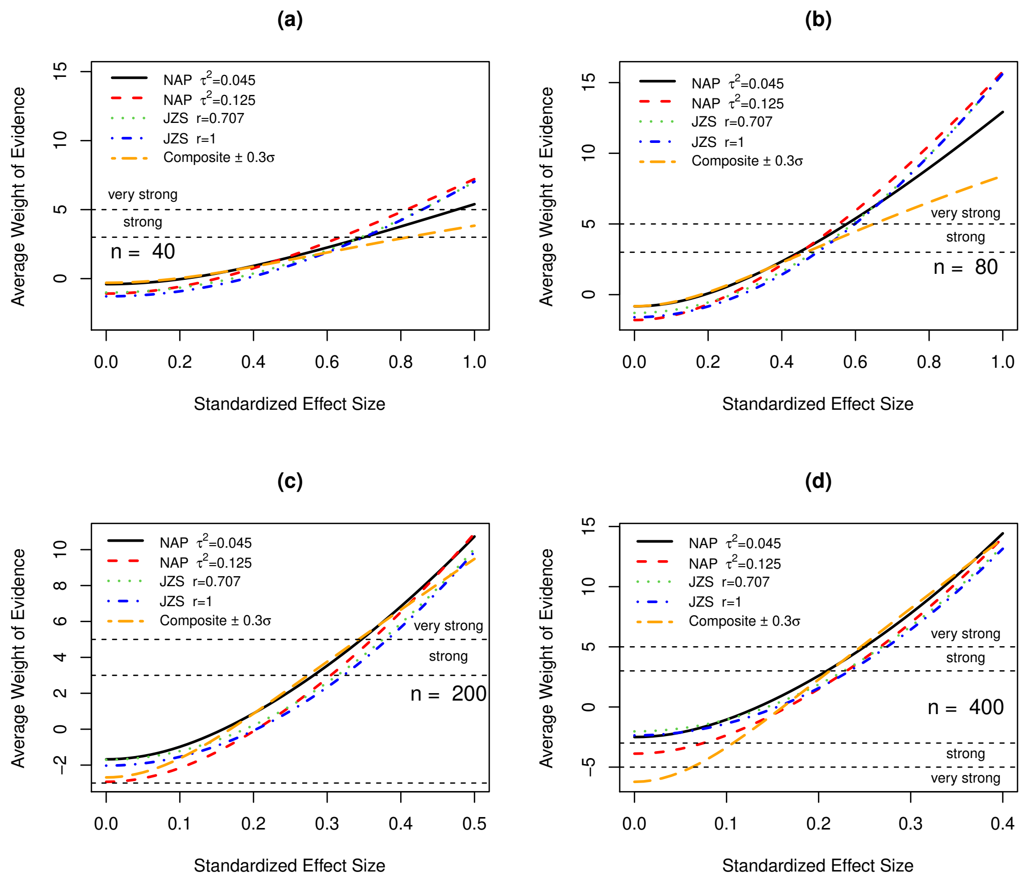
(a)

Figure S2: Weight of evidence for true alternative hypotheses in one-sample z test. Curves depicted in the plots denote the average weight of evidence versus true effect size when the alternative hypothesis was defined by various NAP and JZS densities.



(a)

Figure S3: Weight of evidence for true alternative hypotheses in two-sample z test. Curves depicted in the plots denote the average weight of evidence versus true effect size when the alternative hypothesis was defined by various NAP and JZS densities.



(a)

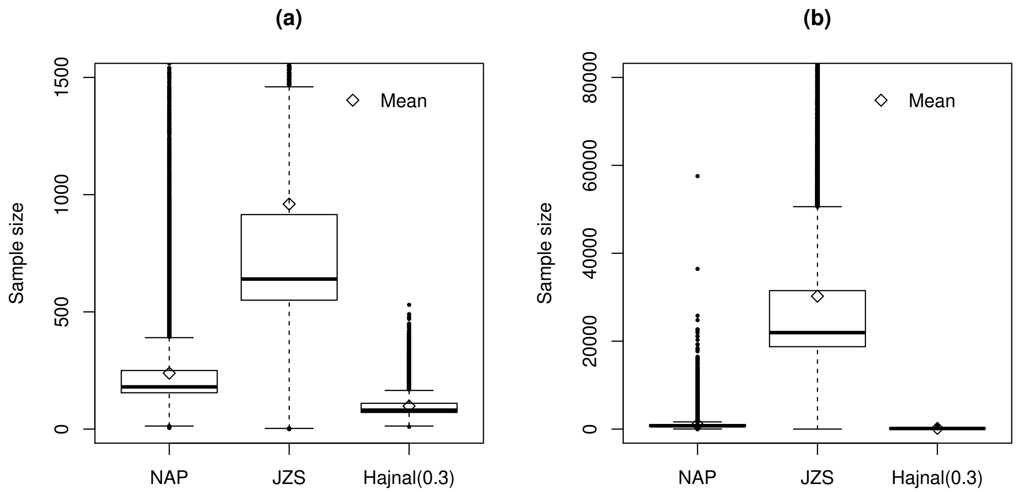
Figure S4: Weight of evidence for true alternative hypotheses in two-sample t test. Curves depicted in the plots denote the average weight of evidence versus true effect size when the alternative hypothesis was defined by various NAP and JZS densities.

S4.2 Sequential tests

Fig. S5–S16 display the operating characteristics of the Hajnal(0.3), default SBF-NAP and default SBF-JZS tests. The results presented below correspond to one-sample z tests and two-sample z and t tests under a true null and alternative hypothesis. For the two-sample tests, equal sample sizes were assumed drawn from both populations. For these tests, the ASN refers to the sample size from each group required on average by the sequential tests. As in the main article, two types of exceedance thresholds were considered: (a) symmetric exceedance thresholds of ± 3 and ± 5, and (b) SPRT thresholds with (α, β) equal to $(0.05, 0.2)$ and $(0.005, 0.05)$. The figures are comparable to Fig. 5–8 in the main article.

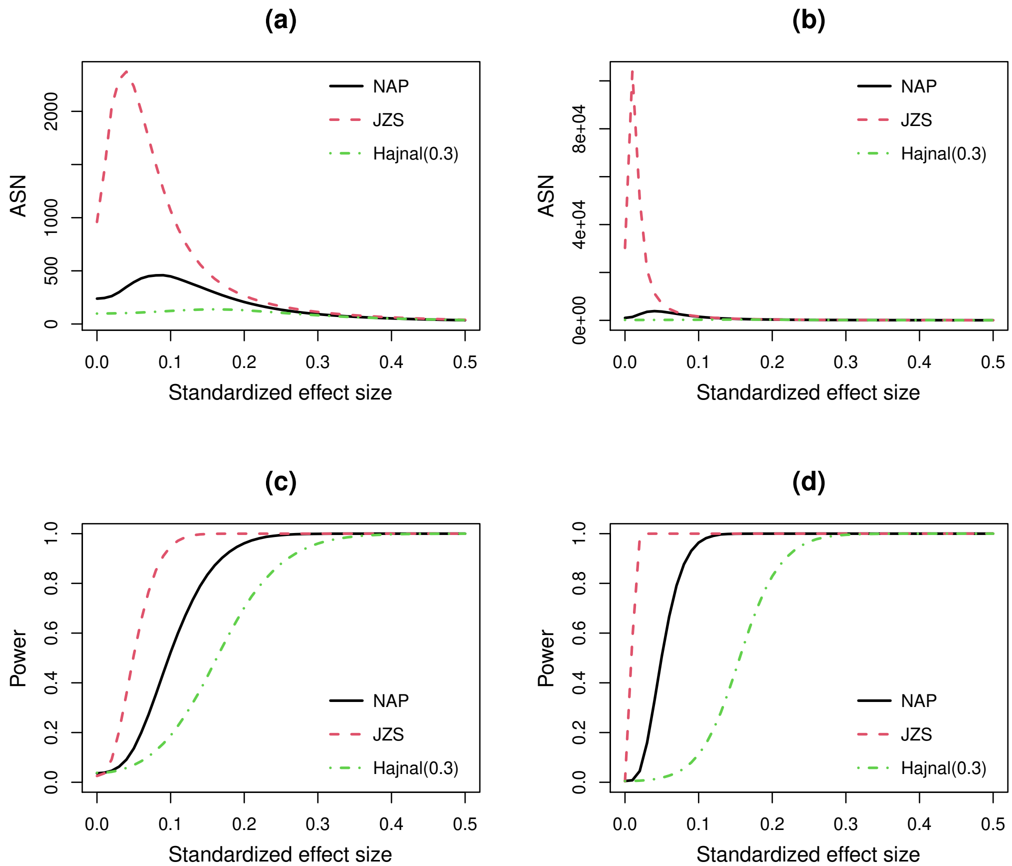
S4.2.1 Performance comparison for symmetric evidence thresholds

Fig. S5–S9 display the operating characteristics of Hajnal(0.3), and the SBF-NAP and SBF-JZS with their default choices. The results presented below correspond to the symmetric exceedance thresholds of ± 3 and ± 5. The figures are comparable to Fig. 5–6 in the main article.



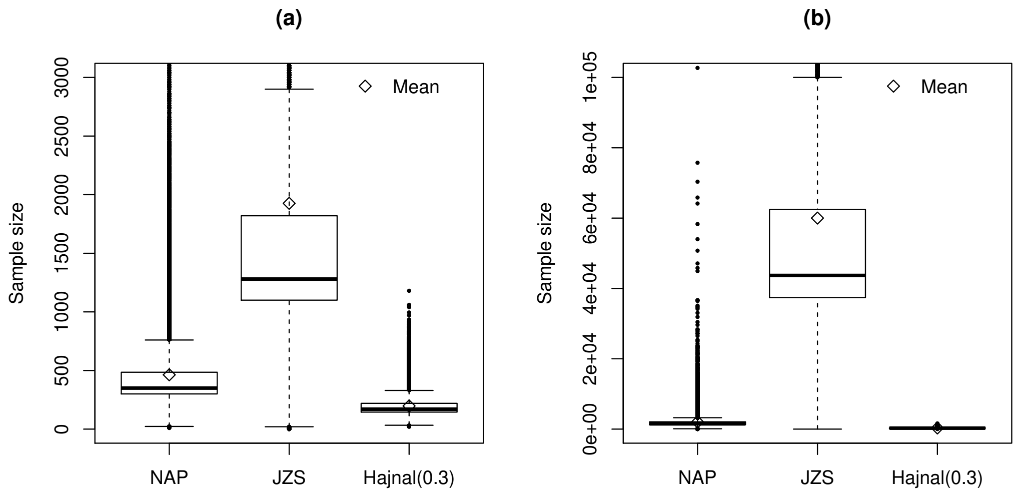
(a)

Figure S5: ASN for sequential procedures under a true null hypothesis in one-sample z test. The plots are truncated at 1500 and 80,000 to enhance comparisons at moderate sample sizes. Panel (a) provides a boxplot estimate of the distribution of sample sizes required for the SBF-NAP, SBF-JZS and Hajnal(0.3) procedures to cross an exceedance threshold of ± 3. About 0.3% percent of SBF-NAP tests and 11% of SBF-JZS tests required more than 1500 samples to reach a decision. All Hajnal(0.3) tests terminated by 530 samples. Panel (b) provides the corresponding boxplots when the exceedance threshold is ± 5. About 4% of SBF-JZS tests required more than 80,000 samples to reach a decision. The black diamonds show the ASN's for each procedure. All SBF-NAP tests reached a decision by 57550 samples, and all Hajnal(0.3) tests terminated by observation 985.



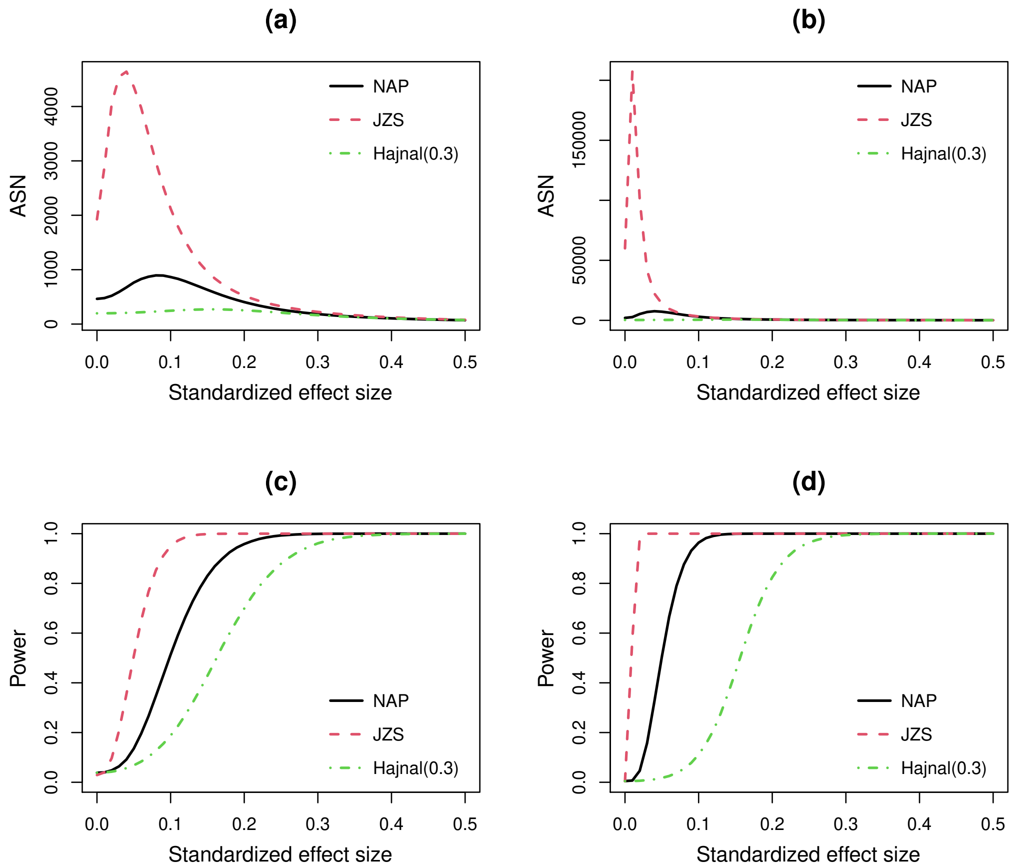
(a)

Figure S6: Operating characteristics under true alternative hypotheses in one-sample z test. Panels (a) and (b) depict the ASN's for three sequential tests when the exceedance thresholds are ± 3 and ± 5, respectively, versus the data-generating value of the standardized effect size. Panels (c) and (d) provide the corresponding probabilities that each test rejects the null hypothesis as a function of the standardized effect size.



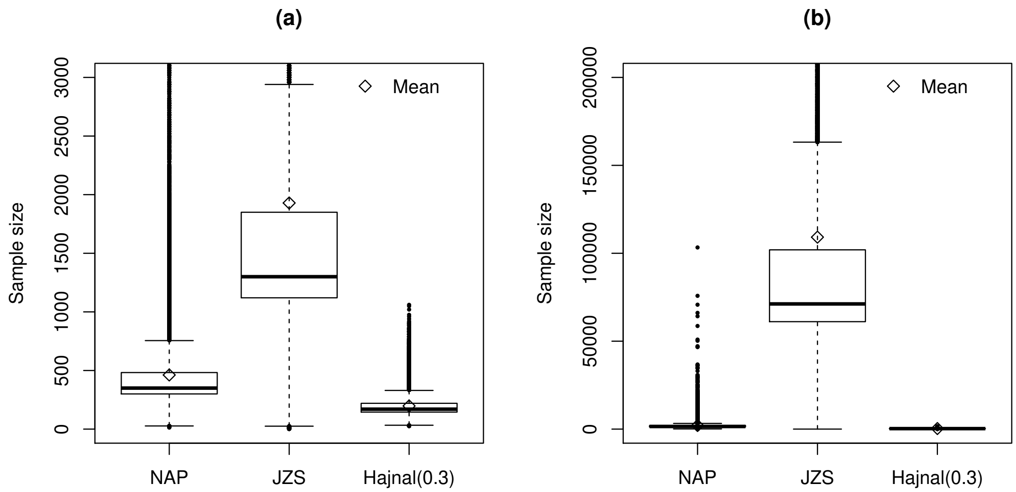
(a)

Figure S7: ASN for sequential procedures under a true null hypothesis in two-sample z test. The plots are truncated at 3000 and 100,000 to enhance comparisons at moderate sample sizes. Panel (a) provides a boxplot estimate of the distribution of sample sizes required from each group for the SBF-NAP, SBF-JZS and Hajnal(0.3) procedures to cross an exceedance threshold of ± 3. About 0.3% of SBF-NAP tests and 11% of SBF-JZS tests required more than 3000 samples from each group to reach a decision. All Hajnal(0.3) tests terminated by 1180 samples. Panel (b) provides the corresponding boxplots when the exceedance threshold is ± 5. About 0.002% of SBF-NAP tests and 10% of SBF-JZS tests required more than 100,000 samples from each group to reach a decision. The black diamonds show the ASN's for each procedure. All Hajnal(0.3) tests terminated by 1600 observations from each group.



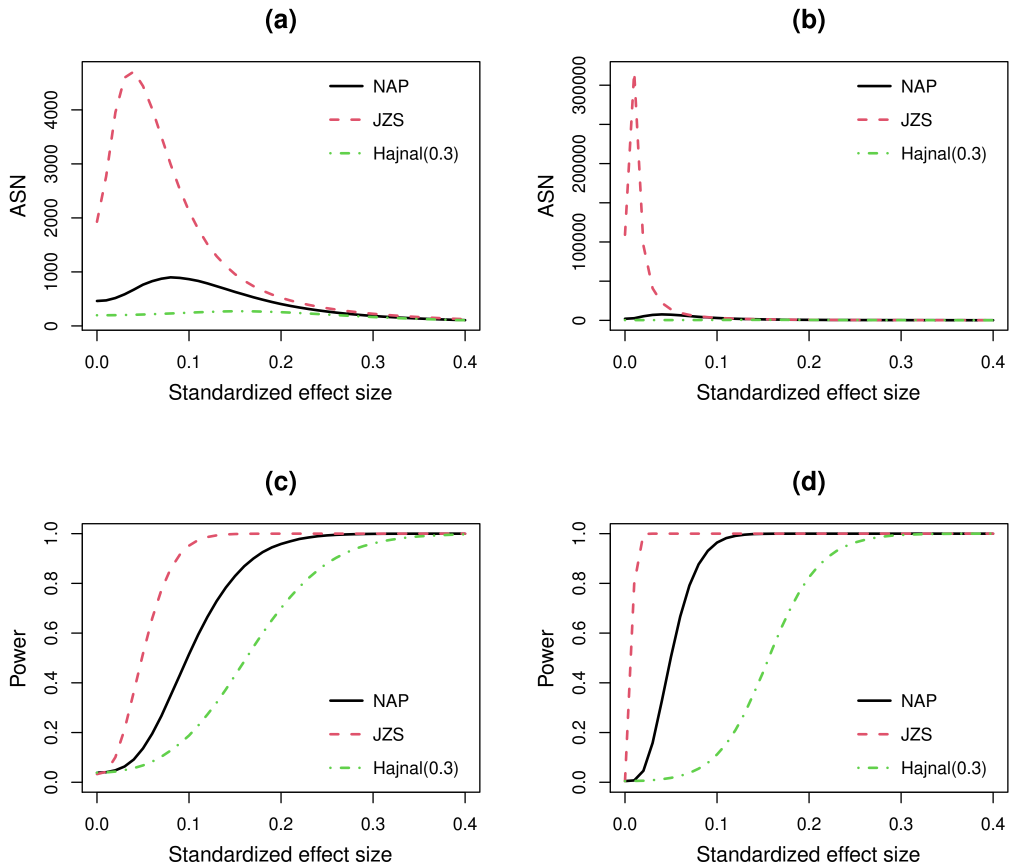
(a)

Figure S8: Operating characteristics under true alternative hypotheses in two-sample z test. Panels (a) and (b) depict the ASN's for three sequential tests when the exceedance thresholds are ± 3 and ± 5, respectively, versus the data-generating value of the standardized effect size. Panels (c) and (d) provide the corresponding probabilities that each test rejects the null hypothesis as a function of the standardized effect size.



(a)

Figure S9: ASN for sequential procedures under a true null hypothesis in two-sample t test. The plots are truncated at 3000 and 200,000 to enhance comparisons at moderate sample sizes. Panel (a) provides a boxplot estimate of the distribution of sample sizes required from each group for the SBF-NAP, SBF-JZS and Hajnal(0.3) procedures to cross an exceedance threshold of ± 3. About 0.3% of SBF-NAP tests and 11% of SBF-JZS tests required more than 3000 samples from each group to reach a decision. All Hajnal(0.3) tests terminated by 1060 samples. Panel (b) provides the corresponding boxplots when the exceedance threshold is ± 5. About 8% of SBF-JZS tests required more than 200,000 samples from each group to reach a decision. The black diamonds show the ASN's for each procedure. All SBF-NAP tests reached a decision by 103300 samples, and all Hajnal(0.3) tests terminated by 1610 samples from each group.

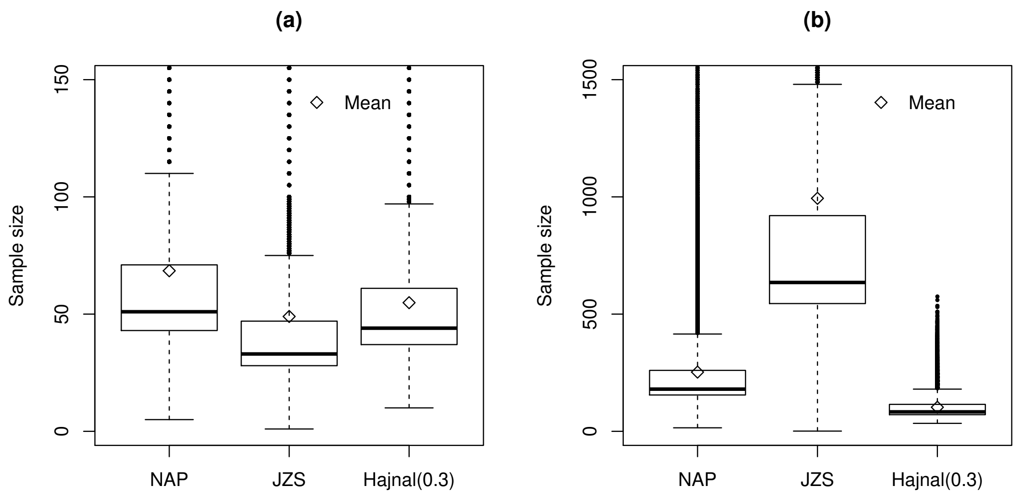


(a)

Figure S10: Operating characteristics under true alternative hypotheses in two-sample t. Panels (a) and (b) depict the ASN's for three sequential tests when the exceedance thresholds are ± 3 and ± 5, respectively, versus the data-generating value of the standardized effect size. Panels (c) and (d) provide the corresponding probabilities that each test rejects the null hypothesis as a function of the standardized effect size.

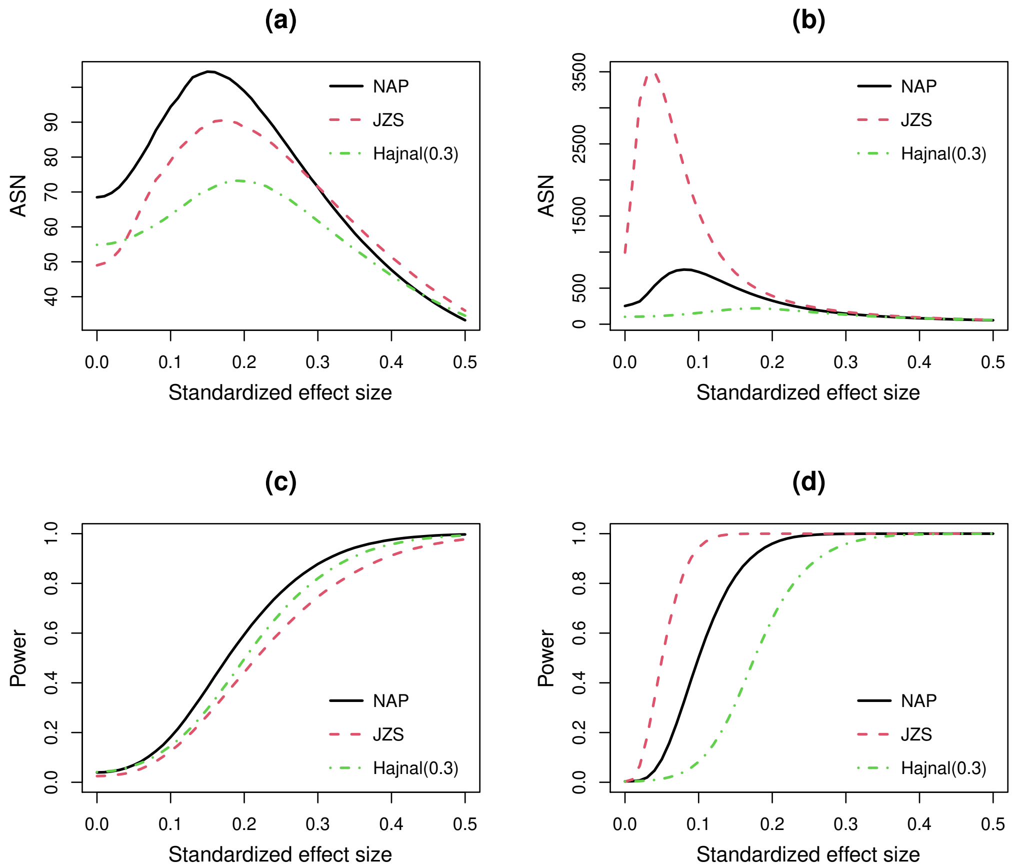
S4.2.2 Performance comparison for the SPRT thresholds

Fig. S11–S16 display the operating characteristics of Hajnal(0.3), default SBF-NAP and default SBF-JZS tests. The results presented below correspond to the SPRT thresholds with (α, β) equal to $(0.05, 0.2)$ and $(0.005, 0.05)$. The figures are comparable to Fig. 7–8 in the main article.



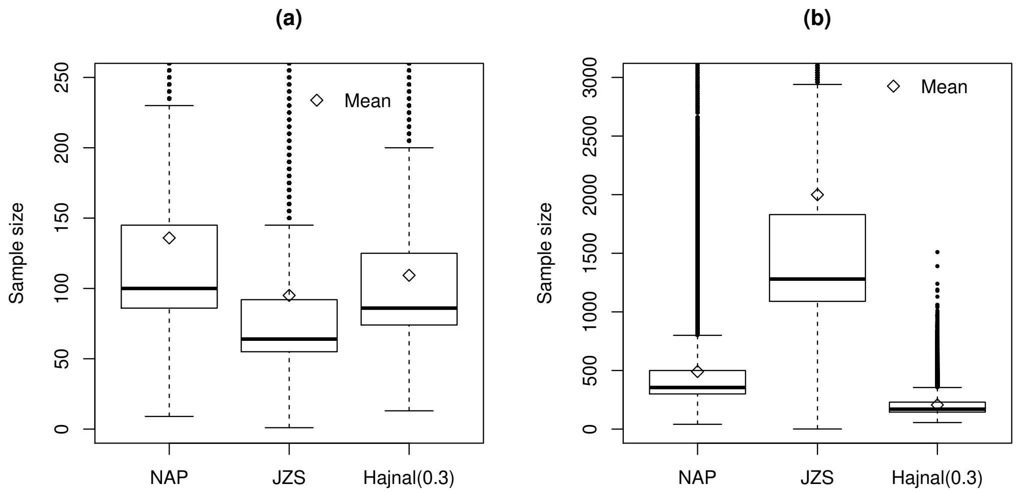
(a)

Figure S11: ASN for SPRT procedures when the null hypothesis is true in one-sample z test. Panel (a) provides a boxplot estimate of the distribution of sample sizes required for the SBF-NAP, SBF-JZS and Hajnal(0.3) procedures to cross Wald's decision thresholds at $\alpha = 0.05$ and $\beta = 0.2$. The plot is truncated at 150 samples (5.3% of SBF-NAP tests, 3.33% of SBF-JZS tests, and 1.71% of Hajnal(0.3) tests required more than 150 samples). Panel (b) provides the corresponding estimate when Wald's decision thresholds were based on $\alpha = 0.005$ and $\beta = 0.05$. The plot is truncated at 1500 samples (0.52% of SBF-NAP and 10.76% of SBF-JZS tests required more than 1500 samples; none of Hajnal(0.3) tests did). The black diamonds show the ASN for each procedure.



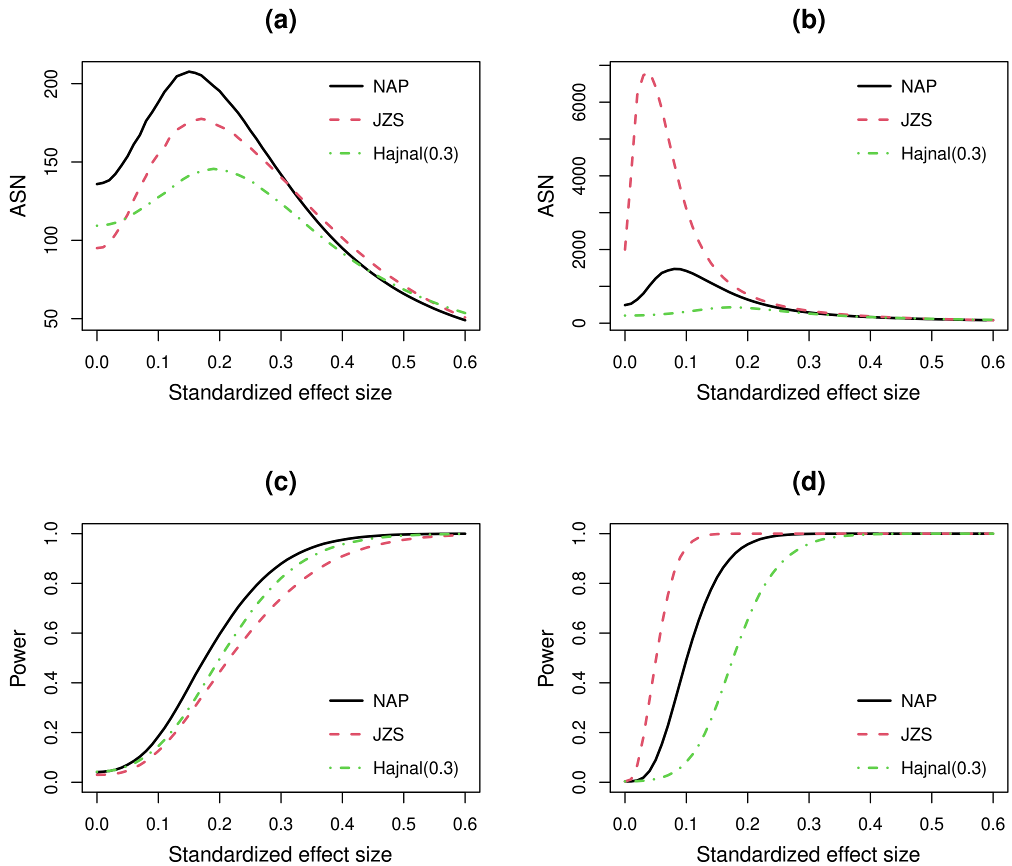
(a)

Figure S12: Operating characteristics under true alternative hypotheses in one-sample z test. Panels (a) and (b) depict the ASN for three SPRT procedures based on Wald's decision thresholds for $(\alpha, \beta) = (0.05, 0.2)$ and $(0.005, 0.05)$, respectively, versus the data-generating value of the standardized effect size. Panels (c) and (d) provide the probability that each procedure rejected the null hypothesis as a function of the standardized effect size.



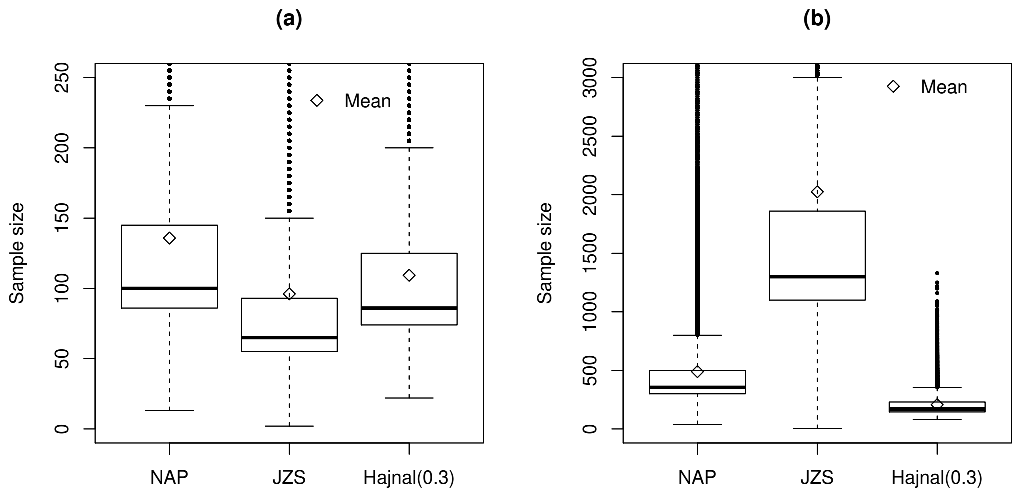
(a)

Figure S13: ASN for SPRT procedures when the null hypothesis is true in two-sample z test. Panel (a) provides a boxplot estimate of the distribution of sample sizes from each group required for the SBF-NAP, SBF-JZS and Hajnal(0.3) procedures to cross Wald's decision thresholds at $\alpha = 0.05$ and $\beta = 0.2$. The plot is truncated at 250 samples (7.68% of SBF-NAP tests, 4.37% of SBF-JZS tests, and 3.3% of Hajnal(0.3) tests required more than 250 samples). Panel (b) provides the corresponding estimate when Wald's decision thresholds were based on $\alpha = 0.005$ and $\beta = 0.05$. The plot is truncated at 3000 samples (0.47% of SBF-NAP and 10.89% of SBF-JZS tests required more than 1500 samples; none of Hajnal(0.3) tests did). The black diamonds show the ASN for each procedure.



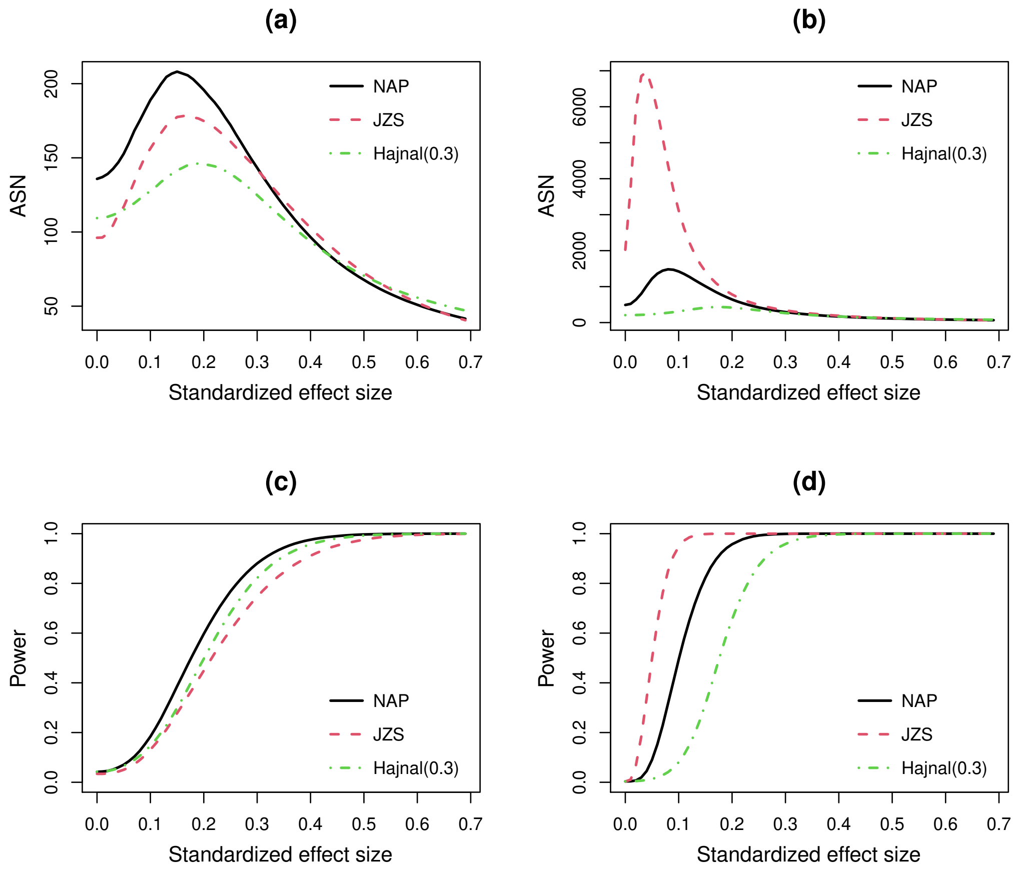
(a)

Figure S14: Operating characteristics under true alternative hypotheses in two-sample z test. Panels (a) and (b) depict the ASN for three SPRT procedures based on Wald's decision thresholds for $(\alpha, \beta) = (0.05, 0.2)$ and $(0.005, 0.05)$, respectively, versus the data-generating value of the standardized effect size. Panels (c) and (d) provide the probability that each procedure rejected the null hypothesis as a function of the standardized effect size.



(a)

Figure S15: ASN for SPRT procedures when the null hypothesis is true in two-sample t test. Panel (a) provides a boxplot estimate of the distribution of sample sizes from each group required for the SBF-NAP, SBF-JZS and Hajnal(0.3) procedures to cross Wald's decision thresholds at $\alpha = 0.05$ and $\beta = 0.2$. The plot is truncated at 250 samples (7.82% of SBF-NAP tests, 4.4% of SBF-JZS tests, and 3.26% of Hajnal(0.3) tests required more than 250 samples). Panel (b) provides the corresponding estimate when Wald's decision thresholds were based on $\alpha = 0.005$ and $\beta = 0.05$. The plot is truncated at 3000 samples (0.47% of SBF-NAP and 11.18% of SBF-JZS tests required more than 1500 samples; none of Hajnal(0.3) tests did). The black diamonds show the ASN for each procedure.



(a)

Figure S16: Operating characteristics under true alternative hypotheses in two-sample t test. Panels (a) and (b) depict the ASN for three SPRT procedures based on Wald's decision thresholds for $(\alpha, \beta) = (0.05, 0.2)$ and $(0.005, 0.05)$, respectively, versus the data-generating value of the standardized effect size. Panels (c) and (d) provide the probability that each procedure rejected the null hypothesis as a function of the standardized effect size.

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