Appendix A

Mean and Variance Expectations of LDS Models

Notation.

μ_0 = Intercept mean	μ_s = Slope mean	
α = Slope loading	β = Auto-proportion	γ = Coupling
σ_0^2 = Intercept variance	σ_s^2 = Slope variance	σ_e^2 = Residual variance
$\sigma_{0,s}$ = Intercept-slope covariance		

A. 1. Mean (*M*) and Variance (*V*) Expectations for a LDS Univariate Model (*T* = 4)

$$E(M) = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_0 + \mu_0\beta + \alpha\mu_s \\ \mu_0 + 2\mu_0\beta + \mu_0\beta^2 + 2\alpha\mu_s + \mu_s\beta\alpha \\ \mu_0 + 3\mu_0\beta + 3\mu_0\beta^2 + \mu_0\beta^3 + 3\alpha\mu_s + 3\mu_s\beta\alpha + \mu_s\beta^2\alpha \end{bmatrix}$$

$$E(V) = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \begin{bmatrix} \sigma_e^2 + \sigma_0^2 \\ \sigma_e^2 + ((1+\beta)\sigma_0^2 + \alpha\sigma_{0,s})(1+\beta) + ((1+\beta)\sigma_{0,s} + \alpha\sigma_s^2)\alpha \\ \sigma_e^2 + ((1+2\beta+\beta^2)\sigma_0^2 + (2\alpha+\beta\alpha)\sigma_{0,s})(1+2\beta+\beta^2) + ((1+2\beta+\beta^2)\sigma_{0,s} + (2\alpha+\beta\alpha)\sigma_s^2)(2\alpha+\beta\alpha) \\ \sigma_e^2 + ((1+3\beta+3\beta^2+\beta^3)\sigma_0^2 + (3\alpha+3\beta\alpha+\beta^2\alpha)\sigma_{0,s})(1+3\beta+3\beta^2+\beta^3) + ((1+3\beta+3\beta^2+\beta^3)\sigma_{0,s} + (3\alpha+3\beta\alpha+\beta^2\alpha)\sigma_s^2)(3\alpha+3\beta\alpha+\beta^2\alpha) \end{bmatrix}$$

Most notably, when the auto-proportion coefficient (β) takes negative values, the variance can decrease over time.

Given estimates from model

$\mu_0 = 0.101$, $\mu_s = 0.875$, $\alpha = 1.0$ (fixed), $\beta = -.228$, $\sigma_0^2 = 1.0$, $\sigma_s^2 = .023$, $\sigma_e^2 = .074$, $\sigma_{0,s} = .108$, the predicted values for means and variances across the four occasions are:

$$E(M) = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \begin{bmatrix} 0.1010 \\ 0.9530 \\ 1.6107 \\ 2.1185 \end{bmatrix} \text{ and } E(V) = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \begin{bmatrix} 1.0740 \\ 0.8597 \\ 0.7295 \\ 0.6500 \end{bmatrix}$$

And given same model estimates for $T = 12$ (as in the data), the predicted values for means and variances across the four occasions are:

$$E(M)=\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12} \end{bmatrix}\begin{bmatrix} 0.1010 \\ 0.9530 \\ 1.6107 \\ 2.1185 \\ 2.5104 \\ 2.8131 \\ 3.0467 \\ 3.2270 \\ 3.3663 \\ 3.4738 \\ 3.5567 \\ 3.6208 \end{bmatrix} \text{ and } E(V)=\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12} \end{bmatrix}\begin{bmatrix} 1.0710 \\ 0.8579 \\ 0.7285 \\ 0.6494 \\ 0.6007 \\ 0.5706 \\ 0.5517 \\ 0.5398 \\ 0.5322 \\ 0.5272 \\ 0.5239 \\ 0.5217 \end{bmatrix}$$

In the bivariate case (i.e., *Reading – Full* model reported in the manuscript), the predicted means and variances for *Reading* generated by these expectations were

$$E(M)=\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12} \end{bmatrix}\begin{bmatrix} 0.007 \\ 1.040 \\ 1.694 \\ 2.149 \\ 2.492 \\ 2.765 \\ 2.991 \\ 3.182 \\ 3.246 \\ 3.487 \\ 3.610 \\ 3.716 \end{bmatrix} \text{ and } E(V)=\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12} \end{bmatrix}\begin{bmatrix} 1.049 \\ 0.710 \\ 0.606 \\ 0.571 \\ 0.559 \\ 0.555 \\ 0.554 \\ 0.554 \\ 0.555 \\ 0.557 \\ 0.558 \\ 0.560 \end{bmatrix}$$

and the sample statistics (i.e., observed means and variances across the 12 data points) to which the model was fitted were

$$M=\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12} \end{bmatrix}\begin{bmatrix} 0.001 \\ 1.088 \\ 1.693 \\ 2.151 \\ 2.458 \\ 2.751 \\ 2.998 \\ 3.170 \\ 3.358 \\ 3.450 \\ 3.539 \\ 3.795 \end{bmatrix} \text{ and } V=\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12} \end{bmatrix}\begin{bmatrix} 0.996 \\ 0.996 \\ 0.772 \\ 0.714 \\ 0.620 \\ 0.610 \\ 0.530 \\ 0.524 \\ 0.474 \\ 0.453 \\ 0.451 \\ 0.414 \end{bmatrix}$$

which led to the following residuals (observed minus expected)

$$M - E(M) = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12} \end{bmatrix} \begin{bmatrix} -0.007 \\ +0.048 \\ -0.001 \\ +0.002 \\ -0.033 \\ -0.014 \\ +0.007 \\ -0.012 \\ +0.012 \\ -0.037 \\ -0.071 \\ +0.080 \end{bmatrix} \text{ and } V - E(V) = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12} \end{bmatrix} \begin{bmatrix} -0.053 \\ +0.286 \\ +0.166 \\ +0.143 \\ +0.061 \\ +0.055 \\ -0.024 \\ -0.030 \\ -0.082 \\ -0.114 \\ -0.108 \\ -0.146 \end{bmatrix}$$

These residuals indicate that the model was able to capture the pattern of increasing means and decreasing variances over time quite well.

A. 2. Mean and Variance Expectations for a LDS Bivariate Model ($T = 4$; Variable Y, similar for variable X)

$$
E(M) = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}
\begin{bmatrix}
\mu_{y0} \\
\mu_{y0} + \mu_{y0}\beta_y + \alpha_y\mu_{ys} + \gamma_y\mu_{x0} \\
\mu_{y0} + 2\mu_{y0}\beta_y + \mu_{y0}\beta_y^2 + \mu_{y0}\gamma_x\gamma_y + 2\alpha_y\mu_{ys} + \mu_{ys}\beta_y\alpha_y + 2\gamma_y\mu_{x0} + \mu_{x0}\beta_y\gamma_y + \mu_{x0}\beta_x\gamma_y + \gamma_y\alpha_y\mu_{xs} \\
\begin{array}{l}
\mu_{y0} + 3\mu_{y0}\beta_y + 3\mu_{y0}\beta_y^2 + \mu_{y0}\beta_y^3 + 3\mu_{y0}\gamma_x\gamma_y + 2\mu_{y0}\gamma_x\beta_y\gamma_y + \mu_{y0}\beta_x\gamma_y\gamma_x + 3\alpha_y\mu_{ys} + \\
3\mu_{ys}\beta_y\alpha_y + \mu_{ys}\beta_y^2\alpha_y + \mu_{ys}\gamma_y\gamma_x\alpha_y + 3\gamma_y\mu_{x0} + 3\mu_{x0}\beta_y\gamma_y + \mu_{x0}\beta_y^2\gamma_y + 3\mu_{x0}\gamma_y\beta_x + \\
\mu_{x0}\beta_x\beta_y\gamma_y + \mu_{x0}\gamma_x\gamma_y^2 + \mu_{x0}\gamma_y\beta_x^2 + 3\gamma_y\alpha_x\mu_{xs} + \mu_{xs}\alpha_x\beta_y\gamma_y + \mu_{xs}\alpha_x\beta_x\gamma_y
\end{array}
\end{bmatrix}
$$

$$
E(S)=\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}
\begin{bmatrix}
\sigma_{ey}^{2}+\sigma_{0y}^{2} \\[2ex]
\begin{array}{l}
\sigma_{ey}^{2}+(\gamma_y\sigma_{x0}^{2}+\alpha_y\sigma_{ys,\,x0}+(1+\beta_y)\sigma_{y0,\,x0})\gamma_y+((1+\beta_y)\sigma_{y0,\,ys}+\alpha_y\sigma_{ys}^{2}+\gamma_y\sigma_{ys,\,x0})\alpha_y+\\
((1+\beta_y)\sigma_{y0}^{2}+\gamma_y\sigma_{y0,\,x0}+\alpha_y\sigma_{y0,\,ys})(1+\beta_y)
\end{array} \\[2ex]
\begin{array}{l}
\sigma_{ey}^{2}+((2\gamma_y+\beta_y\gamma_y+\beta_x\gamma_y)\sigma_{x0}^{2}+\gamma_y\alpha_x\sigma_{x0,\,xs}+(1+2\beta_y+\beta_y^{2}+\gamma_y\gamma_x)\sigma_{y0,\,x0}+(2\alpha_y+\beta_y\alpha_y)\sigma_{ys,\,x0})\\
(2\gamma_y+\beta_y\gamma_y+\beta_x\gamma_y)+(\gamma_y\alpha_x\sigma_{y0,\,xs}+(2\gamma_y+\beta_y\gamma_y+\beta_x\gamma_y)\sigma_{y0,\,x0}+(2\alpha_y+\beta_y\alpha_y)\sigma_{y0,\,ys}+\\
(1+2\beta_y+\beta_y^{2}+\gamma_y\gamma_x)\sigma_{y0}^{2})(1+2\beta_y+\beta_y^{2}+\gamma_y\gamma_x)+((2\alpha_y+\beta_y\alpha_y)\sigma_{ys,\,ys}+\\
(2\gamma_y+\beta_y\gamma_y+\beta_x\gamma_y)\sigma_{x0,\,xs}+\gamma_y\alpha_x\sigma_{xs}^{2}+(1+2\beta_y+\beta_y^{2}+\gamma_y\gamma_x)\sigma_{y0,\,xs})\gamma_y\alpha_x+\\
(\gamma_y\alpha_x\sigma_{ys,\,xs}+(2\gamma_y+\beta_y\gamma_y+\beta_x\gamma_y)\sigma_{ys,\,x0}+(2\alpha_y+\beta_y\alpha_y)\sigma_{ys}^{2}+\\
(1+2\beta_y+\beta_y^{2}+\gamma_y\gamma_x)\sigma_{y0,\,ys})(2\alpha_y+\beta_y\alpha_y)
\end{array} \\[2ex]
\begin{array}{l}
\sigma_{ey}^{2}+((3\alpha_y+\beta_y\alpha_y+\beta_y^{2}\alpha_y+\gamma_y\gamma_x\alpha_y)\sigma_{ys}^{2}+\\
(3\gamma_y+3\beta_y\gamma_y+\beta_y^{2}\gamma_y+3\gamma_y\beta_x+\beta_x\beta_y\gamma_y+\gamma_x\gamma_y^{2}+\gamma_y\beta_x^{2})\sigma_{ys,\,x0}+\\
(3\gamma_y\alpha_x+\alpha_x\beta_y\gamma_y+\alpha_x\beta_x\gamma_y)\sigma_{ys,\,xs}+(1+3\beta_y+3\beta_y^{2}+\beta_y^{3}+3\gamma_x\gamma_y+2\beta_y\gamma_y\gamma_x+\beta_x\gamma_y\gamma_x)\sigma_{y0,\,ys})\\
(3\alpha_y+3\beta_y\alpha_y+\beta_y^{2}\alpha_y+\gamma_y\gamma_x\alpha_y)+((1+3\beta_y+3\beta_y^{2}+\beta_y^{3}+3\gamma_x\gamma_y+2\beta_y\gamma_y\gamma_x+\beta_x\gamma_y\gamma_x)\sigma_{y0}^{2}+\\
(3\gamma_y\alpha_x+\alpha_x\beta_y\gamma_y+\alpha_x\beta_x\gamma_y)\sigma_{y0,\,xs}+(3\gamma_y+3\beta_y\gamma_y+\beta_y^{2}\gamma_y+3\gamma_y\beta_x+\beta_x\beta_y\gamma_y+\gamma_x\gamma_y^{2}+\gamma_y\beta_x^{2})\sigma_{y0,\,x0}+\\
(3\alpha_y+3\beta_y\alpha_y+\beta_y^{2}\alpha_y+\gamma_y\gamma_x\alpha_y)\sigma_{y0,\,ys})(1+3\beta_y+3\beta_y^{2}+\beta_y^{3}+3\gamma_x\gamma_y+2\beta_y\gamma_y\gamma_x+\beta_x\gamma_y\gamma_x)+\\
((3\gamma_y\alpha_x+\alpha_x\beta_y\gamma_y+\alpha_x\beta_x\gamma_y)\sigma_{x0,\,xs}+(3\alpha_y+3\beta_y\alpha_y+\beta_y^{2}\alpha_y+\gamma_y\gamma_x\alpha_y)\sigma_{ys,\,x0}+\\
(3\gamma_y+3\beta_y\gamma_y+\beta_y^{2}\gamma_y+3\gamma_y\beta_x+\beta_x\beta_y\gamma_y+\gamma_x\gamma_y^{2}+\gamma_y\beta_x^{2})\sigma_{x0}^{2}+\\
(1+3\beta_y+3\beta_y^{2}+\beta_y^{3}+3\gamma_x\gamma_y+2\beta_y\gamma_y\gamma_x+\beta_x\gamma_y\gamma_x)\sigma_{y0,\,x0})\\
(3\gamma_y+3\beta_y\gamma_y+\beta_y^{2}\gamma_y+3\gamma_y\beta_x+\beta_x\beta_y\gamma_y+\gamma_x\gamma_y^{2}+\gamma_y\beta_x^{2})+((3\alpha_y+3\beta_y\alpha_y+\beta_y^{2}\alpha_y+\gamma_y\gamma_x\alpha_y)\sigma_{ys,\,ys}+\\
(3\gamma_y+3\beta_y\gamma_y+\beta_y^{2}\gamma_y+3\gamma_y\beta_x+\beta_x\beta_y\gamma_y+\gamma_x\gamma_y^{2}+\gamma_y\beta_x^{2})\sigma_{y0,\,xs}+\\
(3\gamma_y\alpha_x+\alpha_x\beta_y\gamma_y+\alpha_x\beta_x\gamma_y)\sigma_{xs}^{2}+(1+3\beta_y+3\beta_y^{2}+\beta_y^{3}+3\gamma_x\gamma_y+2\beta_y\gamma_y\gamma_x+\beta_x\gamma_y\gamma_x)\sigma_{y0,\,xs})\\
(3\gamma_y\alpha_x+\alpha_x\beta_y\gamma_y+\alpha_x\beta_x\gamma_y)
\end{array}
\end{bmatrix}
$$

Given estimates from model

μ_{y0} = 0.032
μ_{ys} = 1.005
α_y = 1.0 (fixed)
β_y = -0.408
γ_y = 0.087
σ_{y0}^2 = 1.10
σ_{ys}^2 = 0.051
σ_{ey}^2 = 0.069
$\sigma_{y0,ys}$ = 0.165

μ_{x0} = 0.015
μ_{xs} = 0.810
α_x = 1.0 (fixed)
β_x = -0.202
γ_x = 0.229
σ_{x0}^2 = 0.944
σ_{xs}^2 = 0.031
σ_{ex}^2 = 0.176
$\sigma_{x0,xs}$ = 0.091

$\sigma_{y0,x0}$ = 0.602
$\sigma_{y0,xs}$ = -0.006
$\sigma_{ys,x0}$ = 0.089
$\sigma_{ys,xs}$ = -0.010

the predicted values for means and variances across the four occasions for Reading and Full Cognition (i.e., variables *Y* and *X*, respectively) are:

$$E(M_y) = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \begin{bmatrix} 0.032 \\ 1.025 \\ 1.684 \\ 2.150 \end{bmatrix} \text{ and } E(V_y) = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \begin{bmatrix} 1.167 \\ 0.785 \\ 0.644 \\ 0.589 \end{bmatrix}$$

$$E(M_x) = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \begin{bmatrix} 0.015 \\ 0.829 \\ 1.707 \\ 2.557 \end{bmatrix} \text{ and } E(V_x) = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \begin{bmatrix} 1.120 \\ 1.228 \\ 1.300 \\ 1.341 \end{bmatrix}$$

These predicted values show that this bivariate LDS model can capture the pattern of increasing means (for both variables) over time quite well. Moreover, it can also capture the reverse pattern of variances over time for both variables: decreasing for *Reading* and increasing for *Full Cognition*.