Supplementary Material

1. Derivation (including intermediate steps) for Section "Inappropriate Common Factor Models Threaten Validity and Produce Bias"

Let *M* be a unit-weighted composite of *p* components, $m_1 \ldots m_p$, such that $M = \sum_{i=1}^{p} m_i$.

Let *Y* be a variable with unit variance and $\text{cov}\left(m_i, Y\right) = b$ for all m_i. For the sake of simplicity, assume all m_i have unit variance and $\text{cov}\left(m_i, m_j\right) = c$ for all $m_{i \neq j}$. The correlation between *M* and *Y* is thus:

$$\begin{aligned} cor\left(M, Y\right) &= \frac{\text{cov}\left(M, Y\right)}{\sqrt{\text{var}\left(M\right)\text{var}\left(Y\right)}} \\ &= \frac{\text{cov}\left(\sum m_i, Y\right)}{\sqrt{\text{var}\left(\sum m_i\right)(1)}} \\ &= \frac{p\,\text{cov}\left(m_i, Y\right)}{\sqrt{p + p\left(p-1\right)c}} \\ &= \frac{pb}{\sqrt{p + p\left(p-1\right)c}} \end{aligned} \tag{1}$$

When a reflective indicator model is imposed on the set of components m_i such that $m_i = \lambda_i M^* + \varepsilon_i$, this model will perfectly reproduce the covariance matrix Σ_m when $\lambda_i = 1$, $\text{var}\left(M^*\right) = c$, and $\text{var}\left(\varepsilon_i\right) = 1 - c$. The value of $\text{cov}\left(M^*, Y\right)$ that reproduces the covariances between each component m_i and *Y* is *b*. The correlation between M^* and *Y* then becomes:

$$cor\left(M^*, Y\right) = \frac{\text{cov}\left(M^*, Y\right)}{\sqrt{\text{var}\left(M^*\right)\text{var}\left(Y\right)}} = \frac{b}{\sqrt{c}} \tag{2}$$

By dividing Equation (1) by Equation (2), we get an inflation factor that shows how much bigger the latent variable model correlation is compared to the composite variable correlation:

$$\frac{cor\left(M^*,Y\right)}{cor\left(M,Y\right)}=\frac{b}{\sqrt{c}}\cdot\frac{\sqrt{p+p\left(p-1\right)c}}{pb}=\sqrt{\frac{1+\left(p-1\right)c}{pc}} \tag{3}$$

This inflation factor is a function of the number of components, *p*, and the covariance among the components, *c*. Figure 1 (top left) displays the degree of over-estimation as a function of *c* (along the x-axis) and *p* (separate lines).

This result is precisely the opposite of the classic attenuation due to measurement error effect (Spearman, 1904). Whereas the attenuation effect is that using error-laden variables (e.g., a sum score based on a set of congeneric items) rather than reliable variables (e.g., a latent variable indicated by a set of congeneric items) results in under-estimation of correlations, we have just shown that using a latent variable measurement model when a sum score is true results in *misdisattenuation*: Correlations are disattenuated of measurement error that does not exist, producing over-estimation bias.

We can extend this finding to the case in which the components are measured with error. Let $x_1 \ldots x_p$ be unreliable indicators of $m_1 \ldots m_p$, such that $x_i=\sqrt{\rho}m_i+e_i$ and $\text{var}\left(e_i\right)=1-\rho$. Thus, ρ is the reliability of each unit-variance indicator. Let $X=\sum_{i=1}^{p}x_i$ be the composite of these unreliable observed variables. The correlation between *X* and *Y* is thus:

$$\begin{aligned} cor\left(X,Y\right)&=\frac{\text{cov}\left(X,Y\right)}{\sqrt{\text{var}\left(X\right)\text{var}\left(Y\right)}}\\ &=\frac{\text{cov}\left(\sum x_i,Y\right)}{\sqrt{\text{var}\left(\sum x_i\right)\left(1\right)}}\\ &=\frac{p\,\text{cov}\left(x_i,Y\right)}{\sqrt{p+p\left(p-1\right)c\rho}}\\ &=\frac{pb\sqrt{\rho}}{\sqrt{p+p\left(p-1\right)c\rho}} \end{aligned} \tag{4}$$

When a reflective indicator model is imposed on the set of observed variables x_i such that $x_i = \lambda_i M^{**} + \varepsilon_i$, this model will perfectly reproduce the covariance matrix Σ_m when $\lambda_i = 1$, $\text{var}\left(M^*\right) = c\rho$, and $\text{var}\left(\varepsilon_i\right) = 1 - c\rho$. The value of $\text{cov}\left(M^{**}, Y\right)$ that reproduces the covariances between each component x_i and *Y* is $b\sqrt{\rho}$. The correlation between M^{**} and *Y* then becomes:

$$cor\left(M^{**}, Y\right) = \frac{\text{cov}\left(M^{**}, Y\right)}{\sqrt{\text{var}\left(M^{**}\right)\text{var}\left(Y\right)}} = \frac{b\sqrt{\rho}}{\sqrt{\rho c}} = \frac{b}{\sqrt{c}} \tag{5}$$

Note that item reliability drops out of this equation, making it identical to Equation (2). That is, $cor\left(M^{**}, Y\right) = cor\left(M^{*}, Y\right)$. In contrast to its effect on the observed composite score, measurement error does not affect structural correlations in the latent variable model.

By dividing Equation (1) by Equation (4), we get a multiplier that shows how much smaller the unreliable composite correlation is compared to the perfectly reliable composite variable correlation:

$$\begin{aligned} \frac{cor\left(X, Y\right)}{cor\left(M, Y\right)} &= \frac{bp\sqrt{\rho}}{\sqrt{p + p\left(p-1\right)c\rho}} \cdot \frac{\sqrt{p + p\left(p-1\right)c}}{pb} \\ &= \sqrt{\frac{\rho\left[p + p\left(p-1\right)c\right]}{p + p\left(p-1\right)c\rho}} \\ &= \sqrt{\frac{\rho + \left(p-1\right)c\rho}{1 + \left(p-1\right)c\rho}} \end{aligned} \tag{6}$$

This inflation factor is a function of the number of components, *p*, the covariance among the (perfectly reliable) components, *c*, and the reliability, ρ . Figure 1 (right column) displays the degree of under-estimation as a function of *c* and *p* (top-right, holding ρ constant at .8) and ρ and *p* (bottom-right, holding *c* constant at .5). The bottom-left figure displays the degree of over-estimation that occurs when the latent variable model is fit, as a function of ρ (x-axis; as shown in Equation (5) this over-estimation is not affected by ρ) and *p* (separate lines).

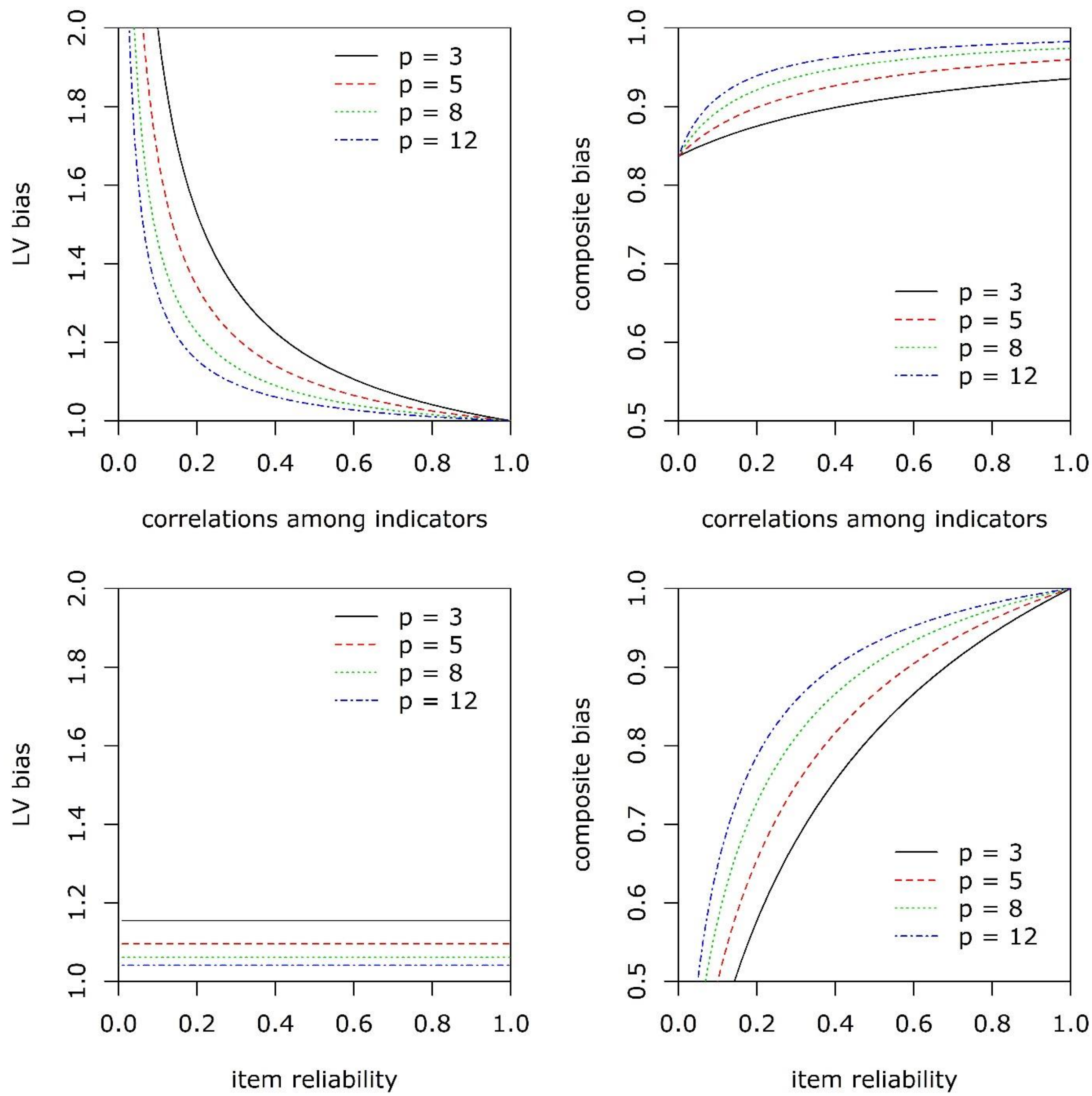

Figure 1. The factor by which a correlation coefficient is over- or under-estimated as a consequence of the model applied when the true construct is defined as a composite of variables that are perfectly measured. Left column: the inflation factor describing the over-estimation of the latent variable correlation as a function of the correlation between indicators, c, the number of indicators, p, and the reliability of the observed measures of each component, ρ *. Right column: the multiplier describing the under-estimation of the observed composite correlation as a function of c, p, and* ρ *. Top row:* ρ *is held constant at .8. Bottom row: c is held constant at .5.*

2. Parameter Generalizations

To assess the generalizability of the results from Examples 1-3, we examined bias and misfit resulting from the full range of possible parameter values for each model under a set of limiting constraints. This supplement presents methodological details and plots for all 3 examples. R code is available on https://osf.io/f8g4d/.

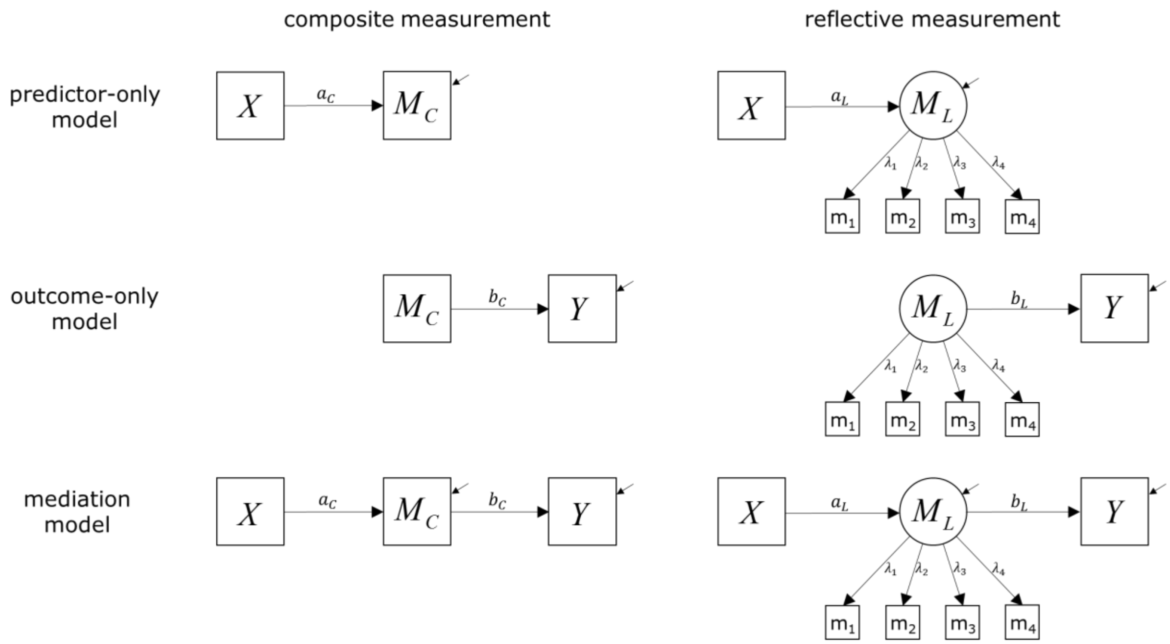

Figure 2. Fitted Models. In the composite measurement models (left column), $m_1 - m_4$ are summed to form M_C. In the common factor measurement models (right column), $m_1 - m_4$ are modeled as reflective indicators of M_L.

R (R Development Core Team, 2016) was used to generate the population covariance matrices. The models shown in Figure 2 were fit to each population matrix using lavaan 0.5-23 (Rosseel, 2012). All R code for generating and fitting models is available at https://osf.io/f8g4d/. We evaluated the performance of the six models according to the accuracy of parameter estimates and model fit.

Accuracy. We examined the accuracy of the model-estimated *a* and *b* paths, which reflect the estimated effect of *X* on *M*, and of *M* on *Y*. Accuracy is assessed by plotting the observed estimates resulting from each model.

Model Fit. We examined model fit to determine whether larger bias in the estimated paths can be detected using fit statistics. We used the population RMSEA, $RMSEA = \sqrt{\hat{F}_{ML}/df}$, to indicate the degree of model misspecification in each generated population.

Example 1

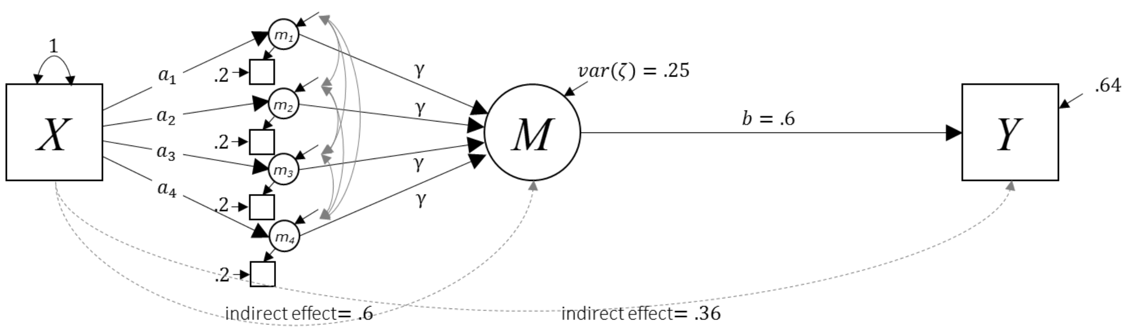

Figure 3. Population model for Example 1. Each set of parameters was randomly drawn, subject to the following constraints: $a_1 - a_4$ path values were drawn from $U(0,.8)$; the value of γ was chosen to ensure that the total indirect effect of X on M was .6; all bivariate correlations among $m_1 - m_4$ were equivalent and chosen so that the total variance of M would be 1.

We fit the 6 models in Figure 2 to 20,000 population covariance matrices. Each population was based on a randomly generated set of values of a_1 - a_4 (i.e., the paths from *X* to each causal indicator, see Figure 3) from $U(0,.8)$. The value of γ (i.e., the path coefficient from each indicator to *M*, which was the same for all indicators) and the correlation between each pair of causal indicators (which was held equal for all pairs) was selected so that the total effect of *X* on the composite *M* was always $a = .6$ and the variances of $m_1 - m_4$ and *M* were 1. The residual covariance among each pair of causal indicators was computed based on the total correlation between each pair of indicators, subtracting the correlation implied by the $a_1 - a_4$ paths. Thus, though the marginal correlation among each pair of indicators was identical, their residual covariances differed as a function of their different $a_1 - a_4$ paths. The regression coefficient of *Y* on *M* was $b = .6$.

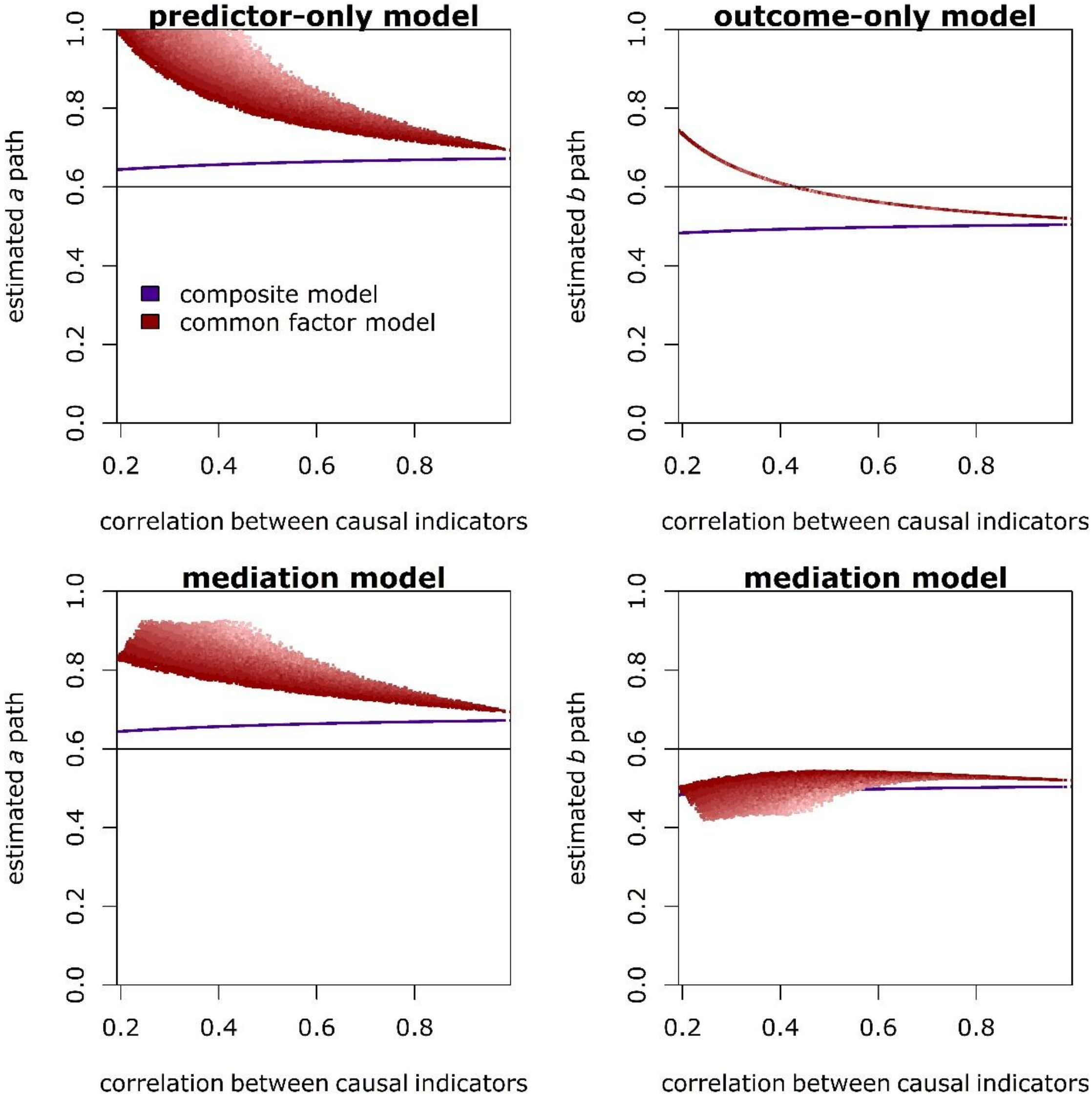

Figure 4. *Standardized coefficients for Example 1 across range of parameter values. Purple points (which appear as a line) correspond to estimates from the composite model. Red points correspond to the common factor model. The dark-to-light red colour gradient indicates the variance in the four* $X \rightarrow m_i$ *path values: Darker indicates more homogeneity in these values. The black line at .6 indicates the parameter value in the true model.*

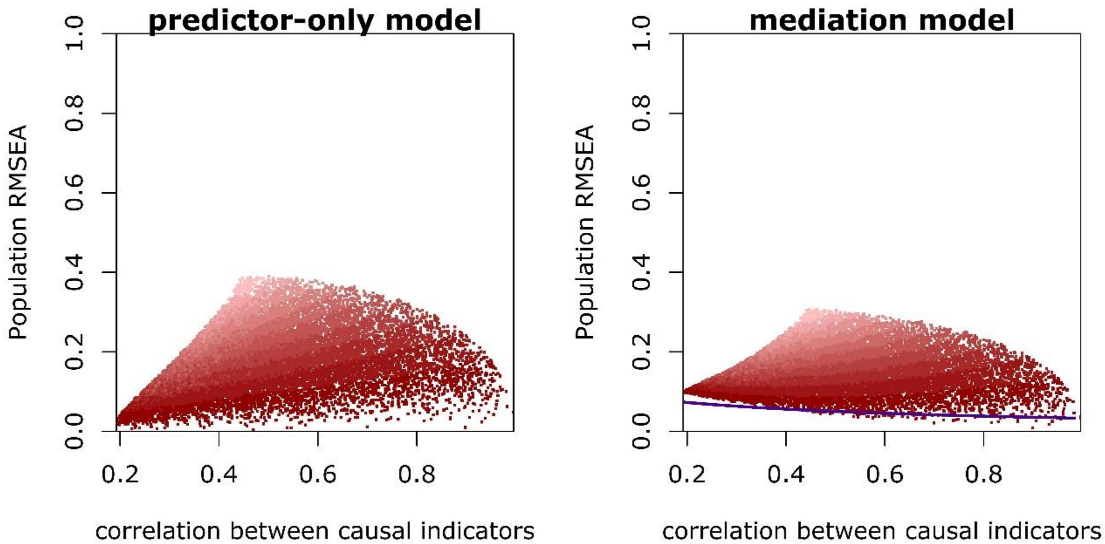

Figure 5. *Example 1 results: Model fit. Red points correspond to the latent variable model and purple points correspond to the composite model. The composite predictor-only and outcome-only models have perfect fit due to having no constrained paths; the latent variable outcome-only models also have perfect fit due to the proportionality constraint being met in these models. The dark-to-light colour gradient indicates the variance in the four $X \rightarrow m_i$ path values: Darker points indicate more homogeneity in these values.*

Example 2

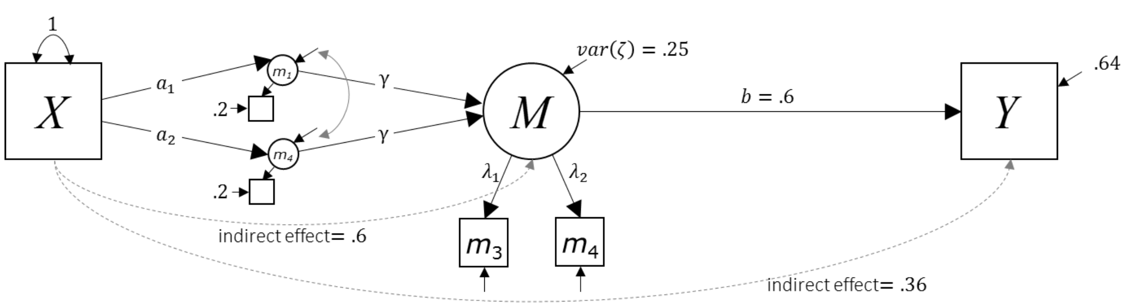

Figure 6. *Population model for Example 2. Each set of parameters was randomly drawn, subject to the following constraints: a_1 and a_2 path values were drawn from $U(0,.8)$; λ_1 and λ_2 values were drawn from $U(.3,1)$, and the value of γ was chosen to ensure that the total indirect effect of X on M was .6; the correlation between m_1 and m_2 was chosen so that the total variance of M would be 1.*

We fit the 6 models in Figure 2 to 20,000 population covariance matrices. The population generating model is shown in Figure 6. We generated random values of a_1 and a_2 (i.e., the paths from *X* to each causal indicator) from $U(.3,.8)$, and values of λ_1 and λ_2 were drawn from $U(.3,1)$. The residual variance of *M* was .25. The value of γ (i.e., the path coefficient from m_1 and m_2 to *M*) and the correlation between m_1 and m_2 was selected so that the total effect of *X* on *M* was always $a = .6$ and the total variance of all variables (except residuals) was 1.

Results. Figure 7 displays the estimated *a* and *b* paths for the six fitted models across the range of parameter values. These results follow a similar pattern to Example 1. As in Example 1, the reflective latent variable model estimates of the *a* path are typically more biased and substantially more variable than those of the composite model.

When the true model is a MIMIC model, both the composite and the latent variable models are wrong, though they are both theoretically appropriate models for two out of the 4 indicators. Perhaps counterintuitively, including two reflective indicators m_3 and m_4 in the composite score actually makes the composite model perform better than in the case where all indicators were causal, because the composite now includes some variance of ζ (i.e., the reflective indicators reflect all of *M*, including ζ). Having a couple of truly reflective indicators also substantially improves the performance of the latent variable model estimates, which are now less biased overall than in the Example 1.

As in the Example 1, the latent variable model estimates – especially the estimate of the *b* path – are affected by whether the model includes *X*, *Y*, or both. The composite model estimates do not change.

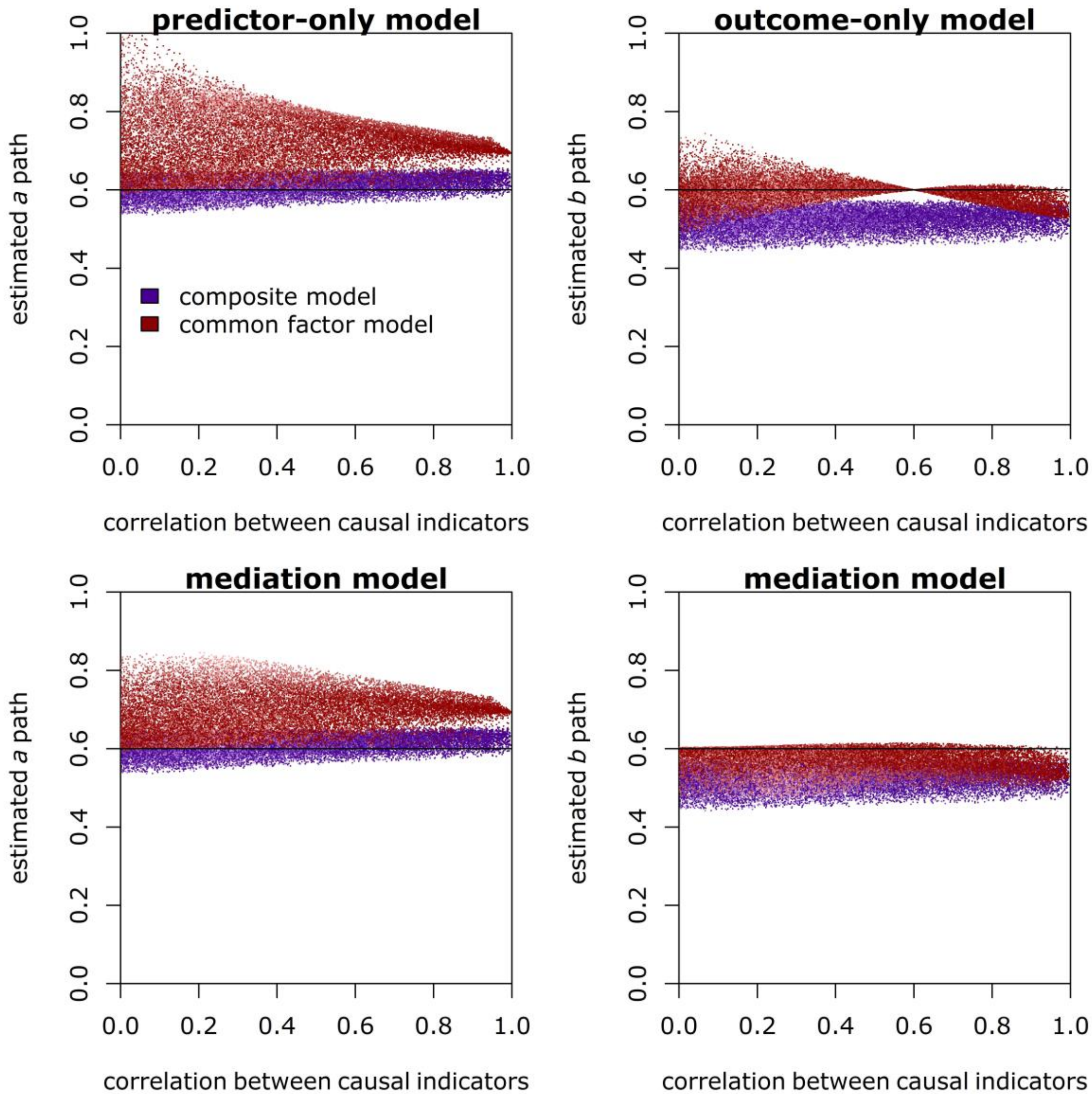

Figure 7. *Example 2 results: Standardized path values. Purple points correspond to estimates from the composite model; red points correspond to the common factor model. The dark-to-light red colour gradient indicates the variance in the two* $X \rightarrow m_i$ *path values: Darker indicates more homogeneity in these values. The black line at .6 indicates the parameter value in the true model.*

Figure 8 displays model fit for all three latent variable models and for the composite mediation model. Compared to Example 1, the latent variable model shows better fit on average, commensurate with its relatively better performance in terms of bias overall. Perfect fit for the outcome-only model when the correlation between the two causal indicators was .6 is a coincidence reflecting the particular way in which model parameter values were selected. The γ path was computed as a function of the correlation between causal indicators so that the total variance of M would be 1. As a result, for those particular sets of parameter values, the outcome-only latent variable model perfectly reproduced the covariances among all four indicators.

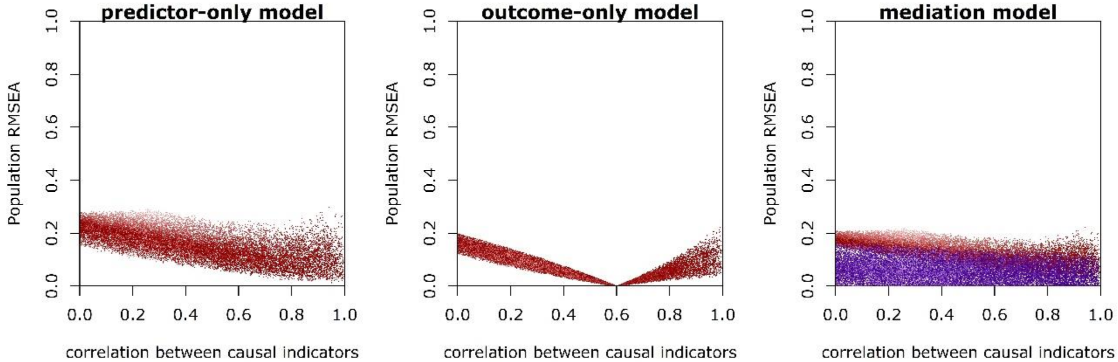

Figure 8. *Example 2 results: Model fit. Red points correspond to the latent variable model and purple points correspond to the composite model. The composite predictor-only and outcome-only models have perfect fit due to having no constrained paths. The dark-to-light colour gradient indicates the variance in the two* $X \rightarrow m_i$ *path values: Darker points indicate more homogeneity in these values.*

Example 3

As we did for Examples 1 and 2, we examined results for a full range of possible parameter values for the model in Example 3, under a set of limiting constraints. The population generating model is shown in Figure 2. We randomly selected sets of values of $\beta_1 - \beta_4$ from U{0, .8}. The value of a' was computed according to the equation $a' = .6/(\beta_1\beta_3 + \beta_2\beta_4)$, so that the total effect of $X \rightarrow m_4$ was .6. This was done to keep the total indirect effect of *X* on *Y* via *M* constant at $.6 \times .6 = .36$, consistent with the previous studies. Note that in this model, it is not possible to evaluate bias in the individual coefficients, $X \rightarrow M$ and $M \rightarrow Y$, because there is no single variable that represents the construct *M*. However, we can examine bias in the indirect effect of *X* on *Y* via *M*, and investigate whether model fit is related to the degree of bias.

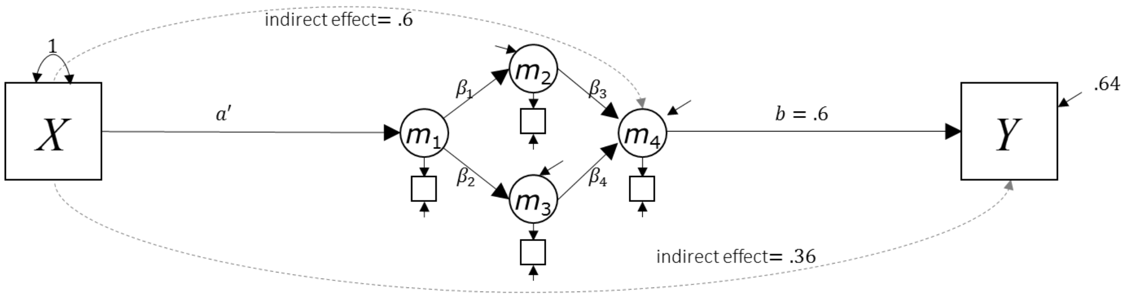

Figure 9. Population model for Example 3. Each set of $\beta_1 - \beta_4$ *parameters were randomly drawn from U{0, .8}. All variables (with the exception of residuals) had variance 1. The value of a′ was selected such that the total effect of* $X \rightarrow m_4$ *was .6 and the indirect effect of X on Y was .36.*

Given the network structure of the true population model, there is no way to choose values of *β* such that a common factor model can describe the covariances among the indicators. Thus, a common factor model of *M* will necessarily fit badly, regardless of the values of the *β* paths and regardless of the presence of *X* or *Y* in the model. The structure of the directed network guarantees that pairs of variables connected by a single arrow (e.g., m_1 and m_2) will share more variance than those connected by two arrows (e.g., m_1 and m_4), and that the conditional independence assumption of the common factor model does not hold (e.g., m_2 and m_3 are independent conditional on m_1). Thus, model misfit results directly from the measurement model, not only when the measurement model is embedded in a larger structural model.

We fit the same six models (composite and common factor predictor-only, outcome-only, and mediation models) as in all previous studies to 20,000 sets of parameter values.

Results. Figure 10 displays the estimated *a* and *b* paths from the six fitted models, as well as the *ab* paths (i.e., the product of the *a* and *b* paths, representing the model-implied mediated effect of *X* on *Y*) from the two meditation models. Along the x-axis of these plots is the value of *a′* (i.e., the $X \rightarrow m_1$ path value; recall that the total effect of *X* on m_4 is always .6, but how that effect is partitioned across the *a′* and the four *β* paths differs for each population matrix). The estimated *a* and *b* paths are interesting to compare across measurement models, but as we noted previously, there is no true parameter value for these paths against which to evaluate bias.

In the predictor-only and outcome-only models, the common factor model tends to produce much higher estimates of *a* and comparable estimates of *b* compared to the composite model. In the mediation model, the common factor model corrects for high *a* values by reducing the value of the estimated *b* path, consistent with previous studies. The common factor mediation model tended to produce the same *a* estimates as the common factor predictor-only model, but substantially lower *b* estimates. This behavior is consistent with the previous examples.

Both the common factor and the composite models estimate the total mediated effect of *X* on *M* to be slightly higher than it is (the black line in the rightmost plot of Figure 10 shows the true mediated path value), and the level of bias in both models is neither very severe nor notably different across the two models.

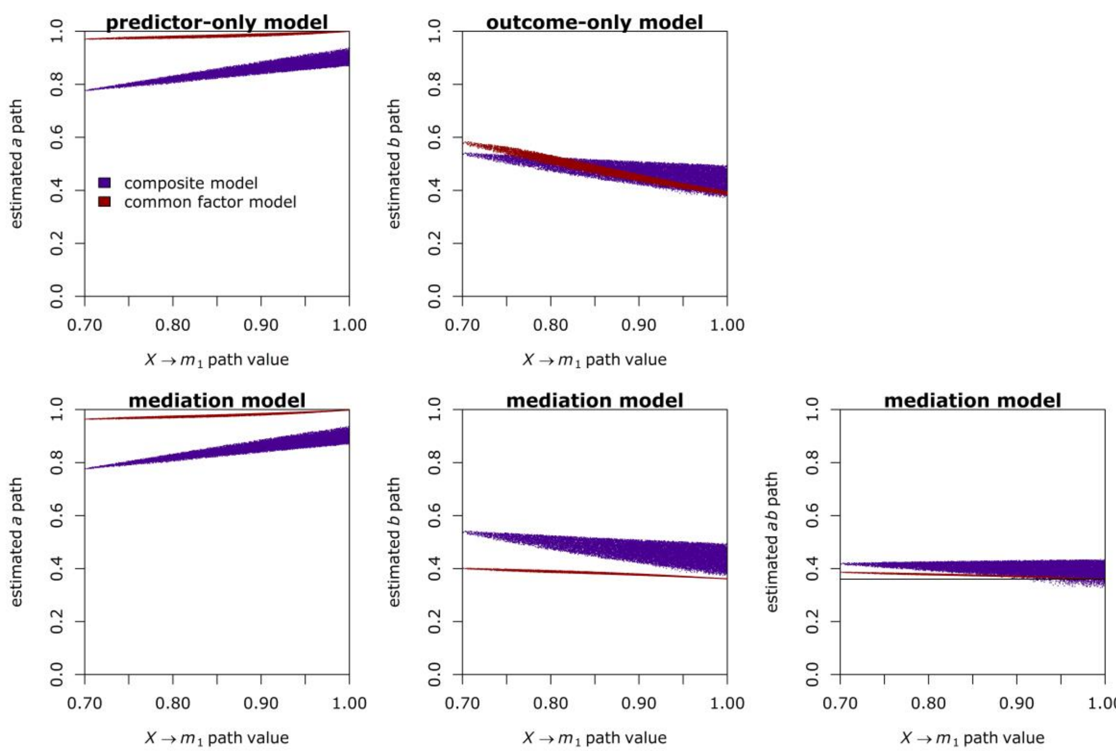

Figure 10. Example 3 results: Standardized path coefficient estimates. Purple points correspond to estimates from the composite model; red points correspond to the common factor model. The four leftmost figures display the estimated a and b path estimates; there is no single true parameter value for these paths. The rightmost figure displays the

product of the estimated a and b paths in the mediation model; the population value for this parameter ($ab = .36$) is indicated by a solid black line.

Figure 11 displays model fit for the three latent variable models and for the composite mediation model. These results confirm that none of the common factor models fit well, reflecting the fact that the common factor measurement model cannot account for either the covariance structure among the network of indicators, nor can it account for the relations between these indicators and external variables. The composite mediation model also fits poorly (though less so), because the association between *X* and *Y* cannot be accounted for as the product of the association between *X* and the sum of m_1-m_4, and between the sum of m_1-m_4 and *Y*.

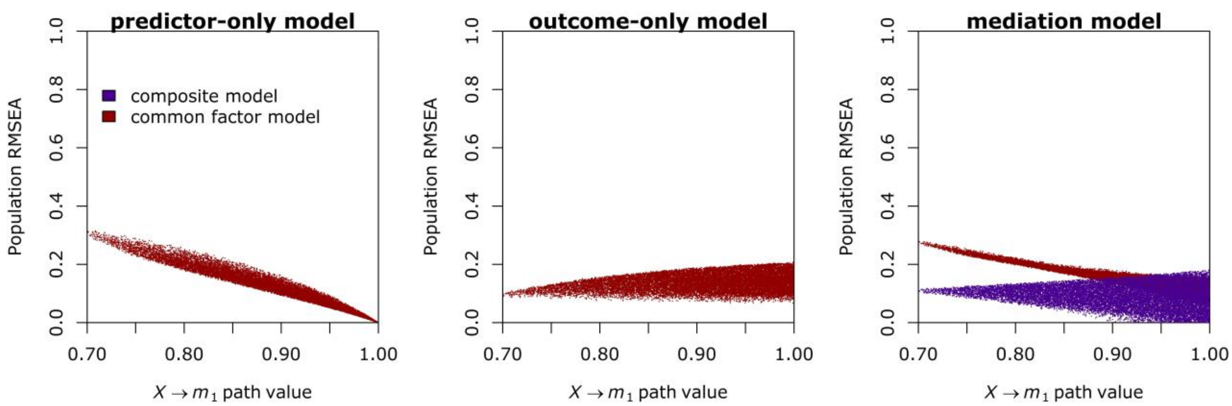

Figure 11. Example 3 results: Model fit. Purple points correspond to estimates from the composite model; Red points correspond to the common factor model.