Online Appendices for:
Modeling Latent Growth With Multiple Indicators: A Comparison of Three Approaches

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Contents

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Appendix A

First-Order Autoregressive Structure Among Latent State Residual Variables

Steyer and Schmitt (1994) were among the first to propose the inclusion of an autoregressive structure in LST models to capture stability beyond what can be accounted for by a common trait factor. Cole, Martin, and Steiger (2005) examined the meaning of autoregressive structures in LST models in detail and noted that it is often appropriate to include autoregressive effects between latent state residual variables to model short-term stability or carry-over effects in these models. In a similar vein, Bollen and Curran (2004) presented so-called autoregressive latent trajectory (ALT) models that include autoregressive paths between observed variables in single-indicator LGC models. Murphy et al. (2011) recently studied the meaning of autocorrelations in the SGM.

A first-order autoregressive structure implies a time-dependence among adjacent latent state residuals, with an additional unpredictable error component δ_t:

$$\zeta_t = \beta_{t(t-1)}\zeta_{(t-1)} + \delta_t, \tag{A.1}$$

where $\beta_{t(t-1)}$ is a real constant and the δ_t variables have a mean of zero and are uncorrelated with each other as well as with all other latent and error variables in the model. Given that we assume a first-order autoregressive structure, the notation for the regression coefficient may be simplified by defining:

$$\beta_t \quad \equiv \quad \beta_{t(t-1)}. \tag{A.2}$$

We illustrate a first-order autoregressive structure graphically for the SGM in Figure A1A, the GSGM in Figure A1B, and the ISGM in Figure A1C.

Including first-order autoregressive effects among the latent state residual variables relaxes the assumption that all of the across-time stability is entirely due to the trait and trait-change process. Instead, part of the (short-term) stability can now be captured by the autoregressive effects, accounting for the fact that adjacent measurements are often more highly correlated than measurements that are further apart in time. In practical applications, it is often common to assume time-invariance of the autoregressive effect. This means that $\beta_t = \beta_s = \beta$ is often assumed to hold for all s, t.

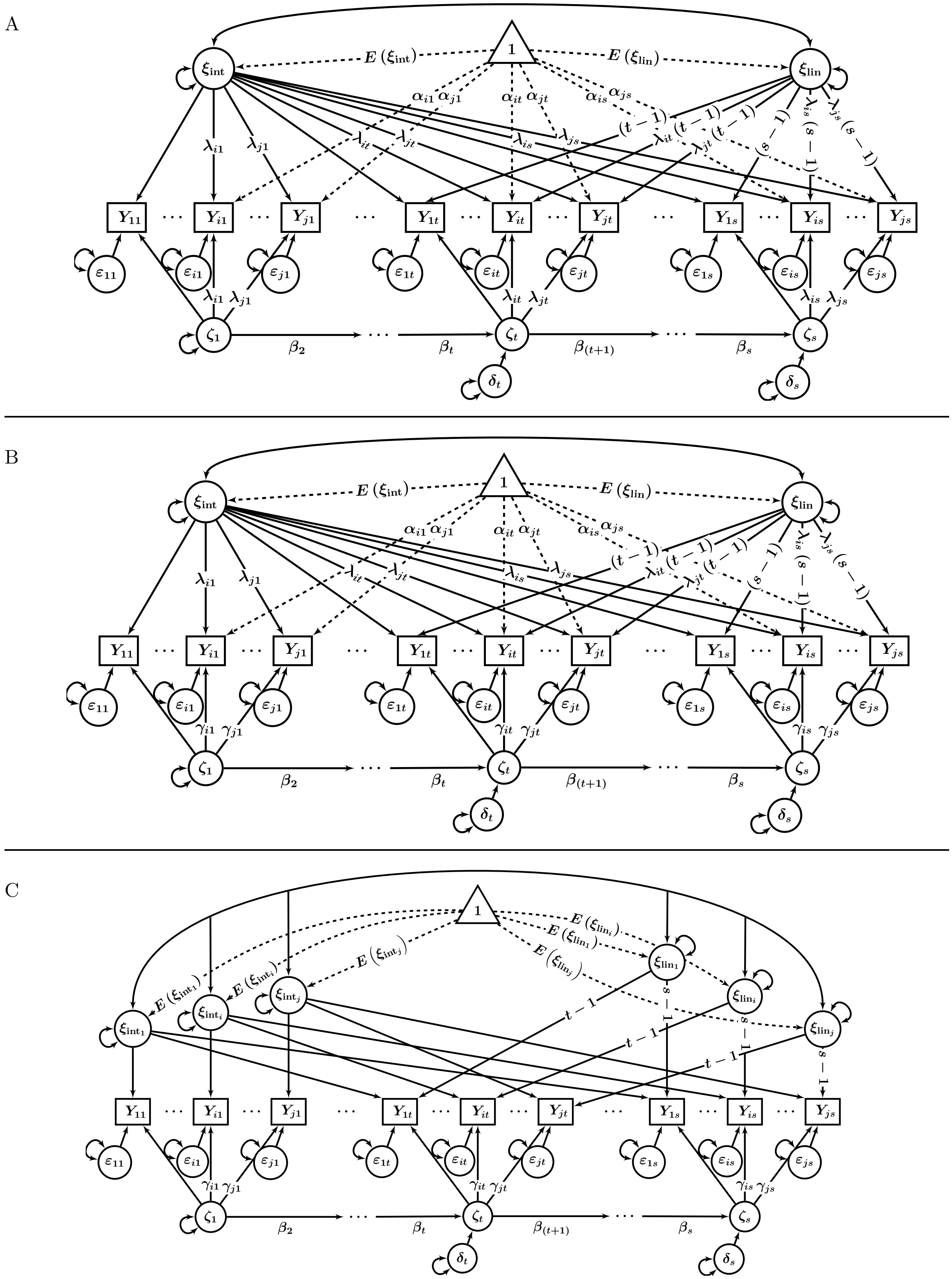


Figure A1. LST Models with First-order Autoregressive Structure Among Latent State Residual Variables A: SGM Post-Transformation with a First-order Autoregressive Structure among Latent State Residual Factors. B: GSGM Post-transformation with a First-order Autoregressive State Residual Structure. C: ISGM Post-transformation with First-order Autoregressive State Residual Structure.

Appendix B
Mean and Covariance Structure in the SGM, GSGM, and ISGM

SGM Mean and Covariance Structure

The combined equations for the SGM are given in Equation (13). Recognizing that we set $\alpha_{1t} = 0$ and $\lambda_{1t} = \gamma_{1t} = 1$ to define the metric of the latent factors, it is sufficient to work with only the final line of these equations. This is given by:

$$Y_{it} = \alpha_{it} + \lambda_{it}\xi_{\text{int}} + \lambda_{it}\left(t-1\right)\xi_{\text{lin}} + \lambda_{it}\zeta_t + \varepsilon_{it}. \tag{B.1}$$

Mean Structure. We derive the mean structure by substituting Equation (B.1) into the expected value, and simplifying this expression according to the algebraic rules for expected values as follows.

$$\begin{aligned} E\left(Y_{it}\right) &= E\left(\alpha_{it} + \lambda_{it}\xi_{\text{int}} + \lambda_{it}\left(t-1\right)\xi_{\text{lin}} + \lambda_{it}\zeta_t + \varepsilon_{it}\right) \\ &= \underbrace{E\left(\alpha_{it}\right)}_{\alpha_{it}} + E\left(\lambda_{it}\xi_{\text{int}}\right) + E\left(\lambda_{it}\left(t-1\right)\xi_{\text{lin}}\right) + \underbrace{E\left(\lambda_{it}\zeta_t\right)}_{0} + \underbrace{E\left(\varepsilon_{it}\right)}_{0} \\ &= \alpha_{it} + \lambda_{it}E\left(\xi_{\text{int}}\right) + \lambda_{it}\left(t-1\right)E\left(\xi_{\text{lin}}\right). \end{aligned}$$

Covariance Structure. First, we expand the covariance by substituting Equation (B.1) into the covariance operator for Y_{it} and Y_{js}:

$$\operatorname{cov}\left(Y_{it}, Y_{js}\right) = \operatorname{cov}\begin{pmatrix} \alpha_{it} + \lambda_{it}\xi_{\text{int}} + \lambda_{it}\left(t-1\right)\xi_{\text{lin}} + \lambda_{it}\zeta_t + \varepsilon_{it}, \\ \alpha_{js} + \lambda_{js}\xi_{\text{int}} + \lambda_{js}\left(s-1\right)\xi_{\text{lin}} + \lambda_{js}\zeta_s + \varepsilon_{js} \end{pmatrix}.$$

Next, we expand using rules of covariance algebra (e.g., Kenny, 1979). This yields

$$\begin{aligned} \operatorname{cov}\left(Y_{it}, Y_{js}\right) &= \underbrace{\operatorname{cov}\left(\alpha_{it},\ \lambda_{js}\left(\xi_{\text{int}} + \left(s-1\right)\xi_{\text{lin}}\right) + \lambda_{js}\zeta_s + \varepsilon_{js}\right)}_{0} + \\ &\quad \underbrace{\operatorname{cov}\left(\lambda_{it}\left(\xi_{\text{int}} + \left(t-1\right)\xi_{\text{lin}}\right) + \lambda_{it}\zeta_t + \varepsilon_{it},\ \alpha_{js}\right)}_{0} + \\ &\quad \operatorname{cov}\begin{pmatrix} \lambda_{it}\left(\xi_{\text{int}} + \left(t-1\right)\xi_{\text{lin}}\right) + \lambda_{it}\zeta_t + \varepsilon_{it}, \\ \lambda_{js}\left(\xi_{\text{int}} + \left(s-1\right)\xi_{\text{lin}}\right) + \lambda_{js}\zeta_s + \varepsilon_{js} \end{pmatrix} \\ &= \operatorname{cov}\begin{pmatrix} \lambda_{it}\left(\xi_{\text{int}} + \left(t-1\right)\xi_{\text{lin}}\right) + \lambda_{it}\zeta_t + \varepsilon_{it}, \\ \lambda_{js}\left(\xi_{\text{int}} + \left(s-1\right)\xi_{\text{lin}}\right) + \lambda_{js}\zeta_s + \varepsilon_{js} \end{pmatrix}. \end{aligned}$$

The first two terms are zero because of the covariance rule $\operatorname{cov}\left(k, X\right) = 0$ where k is a constant. Next, we decompose the covariance according to covariance addition rules. In this step, we separate the error terms ε_{it} and ε_{js}.

$$
\begin{aligned}
\operatorname{cov}(Y_{it}, Y_{js}) &= \operatorname{cov}\begin{pmatrix} \lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}) + \lambda_{it}\zeta_t + \varepsilon_{it}, \\ \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}}) + \lambda_{js}\zeta_s + \varepsilon_{js} \end{pmatrix}, \\
&= \operatorname{cov}\begin{pmatrix} \lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}) + \lambda_{it}\zeta_t + \varepsilon_{it}, \\ \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}}) + \lambda_{js}\zeta_s \end{pmatrix} + \\
&\quad \operatorname{cov}(\lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}) + \lambda_{it}\zeta_t + \varepsilon_{it},\, \varepsilon_{js}), \\
&= \operatorname{cov}\begin{pmatrix} \lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}) + \lambda_{it}\zeta_t + \varepsilon_{it}, \\ \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}}) + \lambda_{js}\zeta_s \end{pmatrix} + \\
&\quad \operatorname{cov}(\lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}),\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\lambda_{it}\zeta_t,\, \varepsilon_{js}) + \operatorname{cov}(\varepsilon_{it},\, \varepsilon_{js}), \\
&= \operatorname{cov}\begin{pmatrix} \lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}) + \lambda_{it}\zeta_t, \\ \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}}) + \lambda_{js}\zeta_s \end{pmatrix} + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}})) + \\
&\quad \operatorname{cov}(\lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}),\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \lambda_{js}\zeta_s) + \operatorname{cov}(\lambda_{it}\zeta_t,\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \varepsilon_{js}).
\end{aligned}
$$

The subsequent step focuses on expanding the first covariance term, the covariance between the latent trait and state residual components.

$$
\begin{aligned}
\operatorname{cov}(Y_{it}, Y_{js}) &= \operatorname{cov}(\lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}),\, \lambda_{js}\zeta_s) + \\
&\quad \operatorname{cov}(\lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}),\, \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}})) + \\
&\quad \operatorname{cov}(\lambda_{it}\zeta_t,\, \lambda_{js}\zeta_s) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}})) + \\
&\quad \operatorname{cov}(\lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}),\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \lambda_{js}\zeta_s) + \operatorname{cov}(\lambda_{it}\zeta_t,\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \varepsilon_{js}).
\end{aligned}
$$

The second term in this expression, which represents the covariance among the latent trait variables is now expanded. At the same time, the constant terms λ_{it}, λ_{js}, $(t-1)$, and $(s-1)$ can be moved outside the covariance expression according to the rule $\operatorname{cov}(k \cdot X, Y) = k \cdot \operatorname{cov}(X, Y)$.

$$\begin{aligned}
\operatorname{cov}\left(Y_{it}, Y_{js}\right) &= \lambda_{it}\lambda_{js}\operatorname{cov}\left(\left(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}\right),\, \zeta_s\right) + \\
&\quad \lambda_{it}\lambda_{js}\begin{pmatrix} [(s-1)+(t-1)]\operatorname{cov}\left(\xi_{\text{int}},\, \xi_{\text{lin}}\right) + \\ (t-1)\,(s-1)\operatorname{var}\left(\xi_{\text{lin}}\right) + \\ \operatorname{var}\left(\xi_{\text{int}}\right) \end{pmatrix} + \\
&\quad \lambda_{it}\lambda_{js}\operatorname{cov}\left(\zeta_t,\, \zeta_s\right) + \\
&\quad \lambda_{js}\operatorname{cov}\left(\varepsilon_{it},\, \left(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}}\right)\right) + \\
&\quad \lambda_{it}\operatorname{cov}\left(\left(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}\right),\, \varepsilon_{js}\right) + \\
&\quad \lambda_{js}\operatorname{cov}\left(\varepsilon_{it},\, \zeta_s\right) + \lambda_{it}\operatorname{cov}\left(\zeta_t,\, \varepsilon_{js}\right) + \\
&\quad \operatorname{cov}\left(\varepsilon_{it},\, \varepsilon_{js}\right).
\end{aligned}$$

Finally, this expression is simplified because for this model, the following restrictions apply

$$\left.\begin{matrix} \operatorname{cov}\left(\xi_{it}, \zeta_t\right) \\ \operatorname{cov}\left(\xi_{it}, \varepsilon_{it}\right) \\ \operatorname{cov}\left(\zeta_t, \varepsilon_{it}\right) \end{matrix}\right\} = 0 \quad \text{for all}\, i, j, s, t, \tag{B.2}$$

$$\operatorname{cov}\left(\zeta_t, \zeta_s\right) = \begin{cases} \operatorname{var}\left(\zeta_t\right) & \text{when}\, t = s, \\ 0 & \text{otherwise,} \end{cases} \tag{B.3}$$

$$\operatorname{cov}\left(\varepsilon_{it}, \varepsilon_{js}\right) = \begin{cases} \operatorname{var}\left(\varepsilon_{it}\right) & \text{when}\, i = j, t = s, \\ 0 & \text{otherwise.} \end{cases} \tag{B.4}$$

The resulting general covariance equation is thus:

$$\begin{aligned}
\operatorname{cov}\left(Y_{it}, Y_{js}\right) &= \lambda_{it}\lambda_{js}\begin{pmatrix} [(s-1)+(t-1)]\operatorname{cov}\left(\xi_{\text{int}},\, \xi_{\text{lin}}\right) + \\ (t-1)\,(s-1)\operatorname{var}\left(\xi_{\text{lin}}\right) + \\ \operatorname{var}\left(\xi_{\text{int}}\right) \end{pmatrix} + \\
&\quad \lambda_{it}\lambda_{js}\operatorname{cov}\left(\zeta_t,\, \zeta_s\right) + \\
&\quad \operatorname{cov}\left(\varepsilon_{it},\, \varepsilon_{js}\right).
\end{aligned}$$

Note that the final two terms are zero except as given in Equations (B.3) and (B.4).

GSGM Mean and Covariance Structure

The combined equations for the GSGM are given in Equation (18). Recognizing that we set $\alpha_{1t} = 0$ and $\lambda_{1t} = \gamma_{1t} = 1$, it is sufficient to work with only the final line of these equations. Thus, the general equation for Y_{it} is given by

$$Y_{it} = \alpha_{it} + \lambda_{it}\xi_{\text{int}} + \lambda_{it}\,(t-1)\,\xi_{\text{lin}} + \gamma_{it}\zeta_t + \varepsilon_{it}. \tag{B.5}$$

Mean Structure. We derive the mean structure by substituting Equation (B.5) into the expected value operator, and simplifying this expression according to the algebraic rules for expected values as follows.

$$
\begin{aligned}
E\left(Y_{it}\right) &= E\left(\alpha_{it}+\lambda_{it}\xi_{\text{int}}+\lambda_{it}\left(t-1\right)\xi_{\text{lin}}+\gamma_{it}\zeta_{t}+\varepsilon_{it}\right) \\
&= \underbrace{E\left(\alpha_{it}\right)}_{\alpha_{it}}+E\left(\lambda_{it}\xi_{\text{int}}\right)+E\left(\lambda_{it}\left(t-1\right)\xi_{\text{lin}}\right)+\underbrace{E\left(\gamma_{it}\zeta_{t}\right)}_{0}+\underbrace{E\left(\varepsilon_{it}\right)}_{0} \\
&= \alpha_{it}+\lambda_{it}E\left(\xi_{\text{int}}\right)+\lambda_{it}\left(t-1\right)E\left(\xi_{\text{lin}}\right).
\end{aligned}
$$

Note that this is equivalent to the mean structure for the SGM, since neither the latent state residual variable ζ_t nor the corresponding loading γ_{it} appears in the final simplified equation.

Covariance Structure. We now derive the covariance structure in a similar way to the SGM. First, we expand the covariance by substituting the above for Y_{it} and Y_{js}:

$$
\text{cov}\left(Y_{it},Y_{js}\right) = \text{cov}\begin{pmatrix} \alpha_{it}+\lambda_{it}\xi_{\text{int}}+\lambda_{it}\left(t-1\right)\xi_{\text{lin}}+\gamma_{it}\zeta_{t}+\varepsilon_{it}, \\ \alpha_{js}+\lambda_{js}\xi_{\text{int}}+\lambda_{js}\left(s-1\right)\xi_{\text{lin}}+\gamma_{js}\zeta_{s}+\varepsilon_{js} \end{pmatrix}.
$$

Next, we expand according to the rules of covariance algebra. This yields

$$
\begin{aligned}
\text{cov}\left(Y_{it},Y_{js}\right) &= \underbrace{\text{cov}\left(\alpha_{it},\, \lambda_{js}\left(\xi_{\text{int}}+\left(s-1\right)\xi_{\text{lin}}\right)+\gamma_{js}\zeta_{s}+\varepsilon_{js}\right)}_{0}+ \\
&\quad \underbrace{\text{cov}\left(\lambda_{it}\left(\xi_{\text{int}}+\left(t-1\right)\xi_{\text{lin}}\right)+\gamma_{it}\zeta_{t}+\varepsilon_{it},\, \alpha_{js}\right)}_{0}+ \\
&\quad \text{cov}\begin{pmatrix} \lambda_{it}\left(\xi_{\text{int}}+\left(t-1\right)\xi_{\text{lin}}\right)+\gamma_{it}\zeta_{t}+\varepsilon_{it}, \\ \lambda_{js}\left(\xi_{\text{int}}+\left(s-1\right)\xi_{\text{lin}}\right)+\gamma_{js}\zeta_{s}+\varepsilon_{js} \end{pmatrix} \\
&= \text{cov}\begin{pmatrix} \lambda_{it}\left(\xi_{\text{int}}+\left(t-1\right)\xi_{\text{lin}}\right)+\gamma_{it}\zeta_{t}+\varepsilon_{it}, \\ \lambda_{js}\left(\xi_{\text{int}}+\left(s-1\right)\xi_{\text{lin}}\right)+\gamma_{js}\zeta_{s}+\varepsilon_{js} \end{pmatrix}.
\end{aligned}
$$

The first two terms are zero because of the covariance rule $\text{cov}\left(k,X\right)=0$ where k is a constant. Next, we decompose the covariance according to the rules of covariance addition. In this step, we separate the error terms ε_{it} and ε_{js}.

$$
\begin{aligned}
\operatorname{cov}(Y_{it}, Y_{js}) &= \operatorname{cov}\begin{pmatrix} \lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}) + \gamma_{it}\zeta_t + \varepsilon_{it}, \\ \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}}) + \gamma_{js}\zeta_s + \varepsilon_{js} \end{pmatrix}, \\
&= \operatorname{cov}\begin{pmatrix} \lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}) + \gamma_{it}\zeta_t + \varepsilon_{it}, \\ \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}}) + \gamma_{js}\zeta_s \end{pmatrix} + \\
&\quad \operatorname{cov}(\lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}) + \gamma_{it}\zeta_t + \varepsilon_{it},\, \varepsilon_{js}), \\
&= \operatorname{cov}\begin{pmatrix} \lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}) + \gamma_{it}\zeta_t + \varepsilon_{it}, \\ \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}}) + \gamma_{js}\zeta_s \end{pmatrix} + \\
&\quad \operatorname{cov}(\lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}),\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\gamma_{it}\zeta_t,\, \varepsilon_{js}) + \operatorname{cov}(\varepsilon_{it},\, \varepsilon_{js}), \\
&= \operatorname{cov}\begin{pmatrix} \lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}) + \gamma_{it}\zeta_t, \\ \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}}) + \gamma_{js}\zeta_s \end{pmatrix} + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}})) + \\
&\quad \operatorname{cov}(\lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}),\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \gamma_{js}\zeta_s) + \operatorname{cov}(\gamma_{it}\zeta_t,\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \varepsilon_{js}).
\end{aligned}
$$

The subsequent step focuses on expanding the first covariance term, the covariance between the latent trait and state residual components.

$$
\begin{aligned}
\operatorname{cov}(Y_{it}, Y_{js}) &= \operatorname{cov}(\lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}),\, \gamma_{js}\zeta_s) + \\
&\quad \operatorname{cov}(\lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}),\, \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}})) + \\
&\quad \operatorname{cov}(\gamma_{it}\zeta_t,\, \gamma_{js}\zeta_s) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \lambda_{js}(\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}})) + \\
&\quad \operatorname{cov}(\lambda_{it}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}}),\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \gamma_{js}\zeta_s) + \operatorname{cov}(\gamma_{it}\zeta_t,\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \varepsilon_{js}).
\end{aligned}
$$

The second term in this expression, which represents the covariance among the latent trait variables is now expanded. At the same time, the constant terms λ_{it}, λ_{js}, γ_{it}, γ_{js}, $(t-1)$, and $(s-1)$ can be moved outside the covariance expression according to the rule $\operatorname{cov}(k \cdot X, Y) = k \cdot \operatorname{cov}(X, Y)$.

$$
\begin{aligned}
\operatorname{cov}(Y_{it}, Y_{js}) \; = \; & \lambda_{it}\gamma_{js} \operatorname{cov}(\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}},\, \zeta_s) + \\
& \lambda_{it}\lambda_{js} \begin{pmatrix} [(s-1)+(t-1)] \operatorname{cov}(\xi_{\text{int}},\, \xi_{\text{lin}}) + \\ (t-1)\,(s-1) \operatorname{var}(\xi_{\text{lin}}) + \\ \operatorname{var}(\xi_{\text{int}}) \end{pmatrix} + \\
& \gamma_{it}\gamma_{js} \operatorname{cov}(\zeta_t,\, \zeta_s) + \\
& \lambda_{js} \operatorname{cov}(\varepsilon_{it},\, (\xi_{\text{int}} + (s-1)\,\xi_{\text{lin}})) + \\
& \lambda_{it} \operatorname{cov}((\xi_{\text{int}} + (t-1)\,\xi_{\text{lin}})\,,\, \varepsilon_{js}) + \\
& \gamma_{js} \operatorname{cov}(\varepsilon_{it},\, \zeta_s) + \gamma_{it} \operatorname{cov}(\zeta_t,\, \varepsilon_{js}) + \\
& \operatorname{cov}(\varepsilon_{it},\, \varepsilon_{js})\,.
\end{aligned}
$$

Finally, this expression is simplified because of the restrictions for this model, which are listed in Equations (B.2), (B.3), and (B.4). The resulting general covariance equation for the GSGM is thus:

$$
\begin{aligned}
\operatorname{cov}(Y_{it}, Y_{js}) \; = \; & \lambda_{it}\lambda_{js} \begin{pmatrix} [(s-1)+(t-1)] \operatorname{cov}(\xi_{\text{int}},\, \xi_{\text{lin}}) + \\ (t-1)\,(s-1) \operatorname{var}(\xi_{\text{lin}}) + \\ \operatorname{var}(\xi_{\text{int}}) \end{pmatrix} + \\
& \gamma_{it}\gamma_{js} \operatorname{cov}(\zeta_t,\, \zeta_s) + \\
& \operatorname{cov}(\varepsilon_{it},\, \varepsilon_{js})\,.
\end{aligned}
$$

As before, the final two terms of this equation are zero except as given in Equations (B.3) and (B.4).

ISGM Mean and Covariance Structure

The combined equations for the ISGM are given in Equation (23). Recognizing that we set $\gamma_{1t} = 1$ to define the metric of each ζ_t, it is sufficient to work with only the final line of these equations. Thus, the general equation for Y_{it} is given by:

$$
Y_{it} = \alpha_{it} + \xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i} + \gamma_{it}\zeta_t + \varepsilon_{it}. \tag{B.6}
$$

Mean Structure. We now derive the covariance structure in a similar way to the SGM and GSGM. First, Equation (B.6) is substituted into the expected value operator, and simplifying this expression according to the algebraic rules for expected values as follows.

$$
\begin{aligned}
E(Y_{it}) &= E(\alpha_{it} + \xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i} + \gamma_{it}\zeta_t + \varepsilon_{it}) \\
&= \underbrace{E(\alpha_{it})}_{\alpha_{it}} + E(\xi_{\text{int}_i}) + E((t-1)\,\xi_{\text{lin}_i}) + \underbrace{E(\gamma_{it}\zeta_t)}_{0} + \underbrace{E(\varepsilon_{it})}_{0} \\
&= \alpha_{it} + E(\xi_{\text{int}_i}) + (t-1)\,E(\xi_{\text{lin}_i})\,.
\end{aligned}
$$

Covariance Structure. The covariance structure for the ISGM is derived in a like manner by substituting Equation (B.6) for Y_{it} and Y_{js} in the covariance operator:

$$\operatorname{cov}(Y_{it}, Y_{js}) = \operatorname{cov}\begin{pmatrix} \xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i} + \gamma_{it}\zeta_t + \varepsilon_{it}, \\ \xi_{\text{int}_j} + (s-1)\,\xi_{\text{lin}_j} + \gamma_{js}\zeta_s + \varepsilon_{js} \end{pmatrix}.$$

Next, we expand according to the covariance algebra rule $\operatorname{cov}(X, Y+Z) = \operatorname{cov}(X,Y) + \operatorname{cov}(X,Z)$. This is used to separate the error terms ε_{it} and ε_{js}.

$$\begin{aligned}
\operatorname{cov}(Y_{it}, Y_{js}) &= \operatorname{cov}\begin{pmatrix} (\xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i}) + \gamma_{it}\zeta_t + \varepsilon_{it}, \\ \left(\xi_{\text{int}_j} + (s-1)\,\xi_{\text{lin}_j}\right) + \gamma_{js}\zeta_s + \varepsilon_{js} \end{pmatrix}, \\
&= \operatorname{cov}\begin{pmatrix} (\xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i}) + \gamma_{it}\zeta_t + \varepsilon_{it}, \\ \left(\xi_{\text{int}_j} + (s-1)\,\xi_{\text{lin}_j}\right) + \gamma_{js}\zeta_s \end{pmatrix} + \\
&\quad \operatorname{cov}((\xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i}) + \gamma_{it}\zeta_t + \varepsilon_{it},\, \varepsilon_{js}), \\
&= \operatorname{cov}\begin{pmatrix} (\xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i}) + \gamma_{it}\zeta_t + \varepsilon_{it}, \\ \left(\xi_{\text{int}_j} + (s-1)\,\xi_{\text{lin}_j}\right) + \gamma_{js}\zeta_s \end{pmatrix} + \\
&\quad \operatorname{cov}(\xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i},\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\gamma_{it}\zeta_t,\, \varepsilon_{js}) + \operatorname{cov}(\varepsilon_{it},\, \varepsilon_{js}), \\
&= \operatorname{cov}\begin{pmatrix} (\xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i}) + \gamma_{it}\zeta_t, \\ \left(\xi_{\text{int}_j} + (s-1)\,\xi_{\text{lin}_j}\right) + \gamma_{js}\zeta_s \end{pmatrix} + \\
&\quad \operatorname{cov}\left(\varepsilon_{it},\, \left(\xi_{\text{int}_j} + (s-1)\,\xi_{\text{lin}_j}\right)\right) + \\
&\quad \operatorname{cov}((\xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i}),\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \gamma_{js}\zeta_s) + \operatorname{cov}(\gamma_{it}\zeta_t,\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \varepsilon_{js}).
\end{aligned}$$

The subsequent step focuses on expanding the first covariance term, the covariance between the latent trait and state residual components.

$$\begin{aligned}
\operatorname{cov}(Y_{it}, Y_{js}) &= \operatorname{cov}(\xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i},\, \gamma_{js}\zeta_s) + \\
&\quad \operatorname{cov}\left(\xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i},\, \xi_{\text{int}_j} + (s-1)\,\xi_{\text{lin}_j}\right) + \\
&\quad \operatorname{cov}(\gamma_{it}\zeta_t,\, \gamma_{js}\zeta_s) + \\
&\quad \operatorname{cov}\left(\varepsilon_{it},\, \xi_{\text{int}_j} + (s-1)\,\xi_{\text{lin}_j}\right) + \\
&\quad \operatorname{cov}(\xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i},\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \gamma_{js}\zeta_s) + \operatorname{cov}(\gamma_{it}\zeta_t,\, \varepsilon_{js}) + \\
&\quad \operatorname{cov}(\varepsilon_{it},\, \varepsilon_{js}).
\end{aligned}$$

The second term in this expression, which represents the covariance among the latent trait variables is now expanded. At the same time, the constant terms γ_{it}, γ_{js}, $(t-1)$, and $(s-1)$ can be moved outside the covariance expression according to the rule $\operatorname{cov}(k \cdot X, Y) = k \cdot \operatorname{cov}(X,Y)$.

$$\begin{aligned}
\operatorname{cov}(Y_{it}, Y_{js}) \quad = \quad & \gamma_{js} \operatorname{cov}\left(\xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i},\, \zeta_s\right) + \\
& \left(\begin{array}{c}
(t-1) \operatorname{cov}\left(\xi_{\text{int}_j},\, \xi_{\text{lin}_i}\right) + \\
(s-1) \operatorname{cov}\left(\xi_{\text{int}_i},\, \xi_{\text{lin}_j}\right) + \\
(t-1)\,(s-1) \operatorname{cov}\left(\xi_{\text{lin}_i}, \xi_{\text{lin}_j}\right) + \\
\operatorname{cov}\left(\xi_{\text{int}_i}, \xi_{\text{int}_j}\right)
\end{array} \right) + \\
& \gamma_{it}\gamma_{js} \operatorname{cov}\left(\zeta_t,\, \zeta_s\right) + \\
& \operatorname{cov}\left(\varepsilon_{it},\, \xi_{\text{int}_j} + (s-1)\,\xi_{\text{lin}_j}\right) + \\
& \operatorname{cov}\left(\xi_{\text{int}_i} + (t-1)\,\xi_{\text{lin}_i},\, \varepsilon_{js}\right) + \\
& \gamma_{js} \operatorname{cov}\left(\varepsilon_{it},\, \zeta_s\right) + \gamma_{it} \operatorname{cov}\left(\zeta_t,\, \varepsilon_{js}\right) + \\
& \operatorname{cov}\left(\varepsilon_{it},\, \varepsilon_{js}\right).
\end{aligned}$$

Finally, this expression is simplified because for this model, the following restrictions apply:

$$\operatorname{cov}(\xi_{it}, \zeta_s) = \operatorname{cov}(\xi_{it}, \varepsilon_{js}) = \operatorname{cov}(\zeta_s, \varepsilon_{it}) = 0,$$

for all i, j, s, and t. The resulting general covariance equation is thus:

$$\begin{aligned}
\operatorname{cov}(Y_{it}, Y_{js}) \quad = \quad & \left(\begin{array}{c}
(t-1) \operatorname{cov}\left(\xi_{\text{int}_j},\, \xi_{\text{lin}_i}\right) + \\
(s-1) \operatorname{cov}\left(\xi_{\text{int}_i},\, \xi_{\text{lin}_j}\right) + \\
(t-1)\,(s-1) \operatorname{cov}\left(\xi_{\text{lin}_i}, \xi_{\text{lin}_j}\right) + \\
\operatorname{cov}\left(\xi_{\text{int}_i}, \xi_{\text{int}_j}\right)
\end{array} \right) + \\
& \gamma_{it}\gamma_{js} \operatorname{cov}\left(\zeta_t,\, \zeta_s\right) + \\
& \operatorname{cov}\left(\varepsilon_{it},\, \varepsilon_{js}\right).
\end{aligned}$$

As before, the final two terms of this equation are zero except as given in Equations (B.3) and (B.4).

Autoregressive (AR) Models

In all three instances, the derivation of the autoregressive (AR) model begins where the corresponding non-AR model left off. This is accomplished by substituting the quantity $\beta_t\zeta_{t-1} + \delta_t$ wherever ζ_t appears. Recall that $\beta_{t(t-1)}$ is a real constant and the δ_t variables have a mean of zero and are uncorrelated with each other as well as with all other latent and error variables in the model. Thus, the mean structure for the AR-model is identical in each case to the mean structure for the non-AR model.

It is not possible to explicitly re-write the equations for the covariance structure because the autoregressive substitution is recursive. Nevertheless, this substitution and expansion may be done for any specific case in which t and s are known. See Appendix E, where we illustrate this expansion for the specific case of two indicators, $i = \{1, 2\}$ and three time points, $t = \{1, 2, 3\}$.

The restrictions for the AR model are slightly different than those of the non-AR model. For the AR model, the following restrictions apply:

$$\left.\begin{array}{c}\operatorname{cov}(\xi_{it},\zeta_s)\\ \operatorname{cov}(\xi_{it},\varepsilon_{js})\\ \operatorname{cov}(\zeta_t,\varepsilon_{js})\\ \operatorname{cov}(\delta_t,\zeta_s)\\ \operatorname{cov}(\xi_{it},\delta_s)\\ \operatorname{cov}(\varepsilon_{it},\delta_s)\end{array}\right\} = 0 \quad \text{for all } i, j, s, t, \tag{B.7}$$

$$\operatorname{cov}(\delta_t,\delta_s) = \begin{cases}\operatorname{var}(\zeta_1) & \text{when } t = s = 1,\\ \operatorname{var}(\delta_t) & \text{when } t = s \neq 1,\\ 0 & \text{otherwise,}\end{cases} \tag{B.8}$$

$$\operatorname{cov}(\varepsilon_{it},\varepsilon_{js}) = \begin{cases}\operatorname{var}(\varepsilon_{it}) & \text{when } i = j, t = s,\\ 0 & \text{otherwise.}\end{cases} \tag{B.9}$$

Summary

The mean and covariance structure for the three models is summarized in Table B1.

Table B1

Mean and Covariance Equations for Non-Autoregressive SGM, GSGM, and ISGM Models

Mean and Covariance Structure			
SGM, non-AR			
Mean	$E(Y_{it})$	=	$\alpha_{it} + \lambda_{it} E(\xi_{\text{int}}) + \lambda_{it}(t-1) E(\xi_{\text{lin}})$
Covariance	$\operatorname{cov}(Y_{it}, Y_{js})$	=	$\lambda_{it}\lambda_{js}[(s-1)+(t-1)] \operatorname{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) +$
			$\lambda_{it}\lambda_{js}(t-1)(s-1) \operatorname{var}(\xi_{\text{lin}}) +$
			$\lambda_{it}\lambda_{js} \operatorname{var}(\xi_{\text{int}}) +$
			$\lambda_{it}\lambda_{js} \operatorname{cov}(\zeta_t, \zeta_s) +$
			$\operatorname{cov}(\varepsilon_{it}, \varepsilon_{js})$
GSGM, non-AR			
Mean	$E(Y_{it})$	=	$\alpha_{it} + \lambda_{it} E(\xi_{\text{int}}) + \lambda_{it}(t-1) E(\xi_{\text{lin}})$
Covariance	$\operatorname{cov}(Y_{it}, Y_{js})$	=	$\lambda_{it}\lambda_{js}[(s-1)+(t-1)] \operatorname{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) +$
			$\lambda_{it}\lambda_{js}(t-1)(s-1) \operatorname{var}(\xi_{\text{lin}}) +$
			$\lambda_{it}\lambda_{js} \operatorname{var}(\xi_{\text{int}}) +$
			$\gamma_{it}\gamma_{js} \operatorname{cov}(\zeta_t, \zeta_s) +$
			$\operatorname{cov}(\varepsilon_{it}, \varepsilon_{js})$
ISGM, non-AR			
Mean	$E(Y_{it})$	=	$E(Y_{it}) = E(\xi_{\text{int}_i}) + (t-1) E(\xi_{\text{lin}_i})$
Covariance	$\operatorname{cov}(Y_{it}, Y_{js})$	=	$(t-1) \operatorname{cov}\left(\xi_{\text{int}_j}, \xi_{\text{lin}_i}\right) +$
			$(s-1) \operatorname{cov}\left(\xi_{\text{int}_i}, \xi_{\text{lin}_j}\right) +$
			$(t-1)(s-1) \operatorname{cov}\left(\xi_{\text{lin}_i}, \xi_{\text{lin}_j}\right) +$
			$\operatorname{cov}\left(\xi_{\text{int}_i}, \xi_{\text{int}_j}\right) +$
			$\gamma_{it}\gamma_{js} \operatorname{cov}(\zeta_t, \zeta_s) +$
			$\operatorname{cov}(\varepsilon_{it}, \varepsilon_{js})$

Appendix C
Proportionality Constraint

The proportionality constraint refers to an implicit constraint on the ratio of general to specific variance that is present in the SGM, but not the GSGM or ISGM. The ratio of general to specific variance for the first time point ($t = 1$) was given in the main text, while this ratio for the general case (for an arbitrary number of time points) is given below. Note that even for the general case, only the ratio for the SGM is subject to the so-called proportionality constraint.

General to Specific Variance Ratio for the SGM

We begin with the decomposition of variance for an observed variable, $\text{var}\,(Y_{it})$, in the SGM, which is given by (the general mean and covariance equations for all three models are derived in Appendix B):

$$\text{var}\,(Y_{it}) = \lambda_{it}^2\,\text{var}\,(\xi_{\text{int}}) + (t-1)^2\,\lambda_{it}^2\,\text{var}\,(\xi_{\text{lin}}) + 2\,(t-1)\,\lambda_{it}^2\,\text{cov}\,(\xi_{\text{int}},\,\xi_{\text{lin}}) + \lambda_{it}^2\,\text{var}\,(\zeta_t) + \text{var}\,(\varepsilon_{it})\,. \tag{C.1}$$

For any two indicators Y_{it} and Y_{jt}, at the same time point t, the variance ratio is constrained to be identical:

$$\begin{aligned} & \frac{\lambda_{it}^2\,\text{var}\,(\xi_{\text{int}}) + (t-1)^2\,\lambda_{it}^2\,\text{var}\,(\xi_{\text{lin}}) + 2\,(t-1)\,\lambda_{it}^2\,\text{cov}\,(\xi_{\text{int}},\,\xi_{\text{lin}})}{\lambda_{it}^2\,\text{var}\,(\zeta_t)} \\ = & \frac{\lambda_{jt}^2\,\text{var}\,(\xi_{\text{int}}) + (t-1)^2\,\lambda_{jt}^2\,\text{var}\,(\xi_{\text{lin}}) + 2\,(t-1)\,\lambda_{jt}^2\,\text{cov}\,(\xi_{\text{int}},\,\xi_{\text{lin}})}{\lambda_{jt}^2\,\text{var}\,(\zeta_t)} \\ = & \frac{\text{var}\,(\xi_{\text{int}}) + (t-1)^2\,\text{var}\,(\xi_{\text{lin}}) + 2\,(t-1)\,\text{cov}\,(\xi_{\text{int}},\,\xi_{\text{lin}})}{\text{var}\,(\zeta_t)}. \end{aligned} \tag{C.2}$$

General to Specific Variance Ratio for the GSGM

In the GSGM, the variance decomposition is given by:

$$\text{var}\,(Y_{it}) = \lambda_{it}^2\,\text{var}\,(\xi_{\text{int}}) + (t-1)^2\,\lambda_{it}^2\,\text{var}\,(\xi_{\text{lin}}) + 2\,(t-1)\,\lambda_{it}^2\,\text{cov}\,(\xi_{\text{int}},\,\xi_{\text{lin}}) + \gamma_{it}^2\,\text{var}\,(\zeta_t) + \text{var}\,(\varepsilon_{it})\,, \tag{C.3}$$

and the ratio of general to specific variance is given by

$$\begin{aligned} & \frac{\lambda_{it}^2\,\text{var}\,(\xi_{\text{int}}) + (t-1)^2\,\lambda_{it}^2\,\text{var}\,(\xi_{\text{lin}}) + 2\,(t-1)\,\lambda_{it}^2\,\text{cov}\,(\xi_{\text{int}},\,\xi_{\text{lin}})}{\gamma_{it}^2\,\text{var}\,(\zeta_t)} \quad \text{for indicator } i, \\ & \frac{\lambda_{jt}^2\,\text{var}\,(\xi_{\text{int}}) + (t-1)^2\,\lambda_{jt}^2\,\text{var}\,(\xi_{\text{lin}}) + 2\,(t-1)\,\lambda_{jt}^2\,\text{cov}\,(\xi_{\text{int}},\,\xi_{\text{lin}})}{\gamma_{jt}^2\,\text{var}\,(\zeta_t)} \quad \text{for indicator } j. \end{aligned} \tag{C.4}$$

The ratio of these variances is not constrained.

General to Specific Variance Ratio for the ISGM

We now consider the general variance equation for the ISGM:

$$\operatorname{var}(Y_{it}) = \operatorname{var}(\xi_{\mathrm{int}_i}) + (t-1)^2 \operatorname{var}(\xi_{\mathrm{lin}_i}) + 2\,(t-1)\operatorname{cov}(\xi_{\mathrm{int}_i},\, \xi_{\mathrm{lin}_i}) + \gamma_{it}^2 \operatorname{var}(\zeta_t) + \operatorname{var}(\varepsilon_{it})\,, \tag{C.5}$$

which leads to the following ratio of general to specific variance:

$$\frac{\operatorname{var}(\xi_{\mathrm{int}_i}) + (t-1)^2 \operatorname{var}(\xi_{\mathrm{lin}_i}) + 2\,(t-1)\operatorname{cov}(\xi_{\mathrm{int}_i},\, \xi_{\mathrm{lin}_i})}{\gamma_{it}^2 \operatorname{var}(\zeta_t)}. \tag{C.6}$$

Obviously, the ISGM also does not impose a proportionality constraint because in general, the state residual loadings γ_{it} in the denominator can be estimated freely for all indicators[1].

[1] Note that there are specific identification conditions in certain designs that may preclude the state residual (γ) loadings from being freely estimated in the GSGM and ISGM. For example, designs with only 2 indicators per time point would not allow the state residual (γ) loadings to be freely estimated unless (1) a significant autoregresive process is present and estimated for the state residual factors or (2) each state residual factor is significantly related to at least one external variable.

Appendix D
Degrees of Freedom in the SGM, GSGM, and ISGM

The degrees of freedom (*df*) of a model is an important consideration when choosing or comparing models; *df* is important when choosing a model because insufficient *df* may result in model non-identification; *df* is also important when comparing models since it provides the context within which fit statistics are interpreted (for more detail see e.g., Walker, 1940).

For simplicity, we compare the *df* for the restricted case in which latent state residuals are assumed to be uncorrelated, and all conditions of MI are met for all models (i.e., that all loadings and intercepts are time-invariant). The *df* for this situation represented as an algebraic function of the input parameters (m and n) are shown in general in Table D3, and for specific numeric values (of m and n) in Table D1. The derivation of the formulas in Table D3 is based on the known and estimated parameters given in Table D2. The technical details for this derivation are given below. Examining these tables reveals that for a given set of variables the SGM has the most *df*, followed by the GSGM, and finally the ISGM.

Computing *df*

The degrees of freedom (*df*) for each model are computed as:

$$df \quad = \quad \#\text{known} - \#\text{estimated}, \tag{D.1}$$

where #known is the number of known parameters (means, variances, and non-redundant covariances of the observed variables Y_{it}) and #estimated is the number of free parameters estimated in the model.

Known Parameters

The total number of known parameters is equal to the number of uniquely identified non-redundant elements in the measurement covariance matrix (including both variances

Table D1
Degrees of Freedom for SGM, GSGM, and ISGM Models

	SGM				GSGM				ISGM			
m	n				n				n			
	3	4	5	6	3	4	5	6	3	4	5	6
2	11	25	43	65	11[a]	25[a]	43[a]	65[a]	4[a]	18[a]	36[a]	58[a]
3	33	65	106	156	31	63	104	154	13	45	86	136
4	64	121	194	283	61	118	191	280	28	85	158	247
5	104	193	307	446	100	189	303	442	48	137	251	390
6	153	281	445	645	148	276	440	640	73	201	365	565

Note. SGM = second-order growth model; GSGM = Generalized second-order growth model; ISGM = indicator-specific growth model; m = number of indicators; n = number of time points. Here, we assume that measurement invariance across time holds for all intercepts and factor loadings. [a]The two indicator GSGM and ISGM are subject to additional constraints needed for model identification.

Table D2

Number of Estimated Model Parameters in Different Multi-indicator Linear Growth Models

Estimated Parameters	Model: SGM	Model: GSGM	Model: ISGM
Intercept and Growth Factor Means			
$E(\xi_{\text{int}_i})$, $E(\xi_{\text{lin}_i})$	2	2	$2m$
Intercept and Growth Factor Variances			
$\text{var}(\xi_{\text{int}_i})$, $\text{var}(\xi_{\text{lin}_i})$	2	2	$2m$
Error Variances			
$\text{var}(\varepsilon_{it})$	mn	mn	mn
State Residual Variances			
$\text{var}(\zeta_t)$ or $\text{var}(\delta_t)$	n	n	n
Intercept and Growth Factor Covariances			
$\text{cov}\left(\xi_{\text{int}_i}, \xi_{\text{int}_j}\right)$, $\text{cov}\left(\xi_{\text{int}_i}, \xi_{\text{lin}_j}\right)$, $\text{cov}\left(\xi_{\text{lin}_i}, \xi_{\text{lin}_j}\right)$	1	1	$m(2m-1)$
Intercepts			
α_{it}	$n(m-1)$ if $\alpha_{it} \neq \alpha_{is}$ $(m-1)$ if $\alpha_{it} = \alpha_{is}$		–
Factor Loadings			
λ_{it}	$n(m-1)$ if $\lambda_{it} \neq \lambda_{is}$ $(m-1)$ if $\lambda_{it} = \lambda_{is}$		–
Autoregressive Effects			
β_t	0 if $\text{cov}(\zeta_t, \zeta_s) = 0$ 1 if $\zeta_t = \beta_t \zeta_{(t-1)} + \delta_t$ and $\beta_t = \beta_s$ $(n-1)$ if $\zeta_t = \beta_t \zeta_{(t-1)} + \delta_{it}$ and $\beta_t \neq \beta_s$		
State Residual Factor Loadings			
γ_{it}	–	$n(m-1)$ if $\gamma_{it} \neq \gamma_{is}$ $(m-1)$ if $\gamma_{it} = \gamma_{is}$	

Note. SGM = second-order growth model; GSGM = Generalized second-order growth model; ISGM = indicator-specific growth model; m = number of indicators; n = number of time points.

Table D3

Algebraic Degrees of Freedom for SGM, GSGM, and ISGM Models

Model	#known	#estimated	*df*
SGM	$\frac{(mn)^2+3mn}{2}$	$2m+mn+n+3$	$\frac{m^2n^2+mn-4m-2n-6}{2}$
GSGM	$\frac{(mn)^2+3mn}{2}$	$mn+3m+n+2$	$\frac{m^2n^2+mn-6m-2n-4}{2}$
ISGM	$\frac{(mn)^2+3mn}{2}$	$2m^2+4m+mn+n-1$	$\frac{m^2n^2-4m^2+mn-8m-2n+2}{2}$

Note. #known = number of known means, variances, and covariances; #estimated = number of estimated parameters; SGM = second-order growth model; GSGM = Generalized second-order growth model; ISGM = indicator-specific growth model; m = number of indicators; n = number of time points. Here, we assume that all latent state residual factors are uncorrelated and that measurement invariance across time holds for all intercepts and factor loadings.

and covariances) plus the total number of expected values that can be computed from the measurements (see, e.g., Bollen, 1989). In the present context, the number of known parameters is

$$\#\text{known} = \frac{(mn)^2+3mn}{2}.$$

Estimated Parameters and Degrees of Freedom

In contrast to the number of known parameters, the number of parameters estimated varies across models. The number of degrees of freedom is then calculated by subtracting the number of estimated parameters from the number of known parameters, as given in Equation (D.1). For simplicity, we assume linear growth, and that all intercepts (α_{it}), trait loadings (λ_{it}), and state residual loadings (γ_{it}) are time-invariant. We also assume that the state residual factors are all uncorrelated (no autoregressive structure). For less restrictive models, or models with other forms of growth, more parameters will be estimated.

SGM.

$$\begin{aligned}\#\text{estimated} &= \underbrace{2}_{\substack{E(\xi_{\text{int}})\\ E(\xi_{\text{lin}})}} + \underbrace{2}_{\substack{\text{var}(\xi_{\text{int}})\\ \text{var}(\xi_{\text{lin}})}} + \underbrace{m\cdot n}_{\text{var}(\varepsilon_{it})} + \underbrace{n}_{\text{var}(\zeta_t)} + \\ &\quad \underbrace{1}_{\text{cov}(\xi_{\text{int}},\xi_{\text{lin}})} + \underbrace{(m-1)}_{\alpha_i} + \underbrace{(m-1)}_{\lambda_i} \\ &= 4+2(m-1)+m\cdot n+n+1 \\ &= 2m+mn+n+3 \\ &= \frac{4m+2mn+2n+6}{2}\end{aligned}$$

Thus, the degrees of freedom for the SGM are given by

$$\begin{aligned}
df &= \#\text{known} - \#\text{estimated} \\
&= \frac{(mn)^2 + 3mn}{2} - \left(\frac{4m + 2mn + 2n + 6}{2}\right) \\
&= \frac{m^2n^2 + 3mn - 4m - 2mn - 2n - 6}{2} \\
&= \frac{m^2n^2 + mn - 4m - 2n - 6}{2}
\end{aligned}$$

GSGM.

$$\begin{aligned}
\#\text{estimated} &= \underbrace{2}_{\substack{E(\xi_{\text{int}}) \\ E(\xi_{\text{lin}})}} + \underbrace{2}_{\substack{\text{var}(\xi_{\text{int}}) \\ \text{var}(\xi_{\text{lin}})}} + \underbrace{mn}_{\text{var}(\varepsilon_{it})} + \underbrace{n}_{\text{var}(\zeta_t)} + \\
&\quad \underbrace{1}_{\text{cov}(\xi_{\text{int}},\xi_{\text{lin}})} + \underbrace{(m-1)}_{\alpha_i} + \underbrace{(m-1)}_{\lambda_i} + \underbrace{(m-1)}_{\gamma_i} \\
&= 4 + 3(m-1) + m \cdot n + n + 1 \\
&= 4 + 3m - 3 + m \cdot n + n + 1 \\
&= mn + 3m + n + 2 \\
&= \frac{2mn + 6m + 2n + 4}{2}
\end{aligned}$$

Thus, the degrees of freedom for the GSGM are given by

$$\begin{aligned}
df &= \#\text{known} - \#\text{estimated} \\
&= \frac{(mn)^2 + 3mn}{2} - \left(\frac{2m \cdot n + 6m + 2n + 4}{2}\right) \\
&= \frac{m^2n^2 + 3mn - 2mn - 6m - 2n - 4}{2} \\
&= \frac{m^2n^2 + mn - 6m - 2n - 4}{2}
\end{aligned}$$

ISGM.

$$\begin{aligned}
\#unknown &= \underbrace{2m}_{\substack{E(\xi_{\text{int}_i}) \\ E(\xi_{\text{lin}_i})}} + \underbrace{2m}_{\substack{\text{var}(\xi_{\text{int}_i}) \\ \text{var}(\xi_{\text{lin}_i})}} + \underbrace{m \cdot n}_{\text{var}(\varepsilon_{it})} + \underbrace{n}_{\text{var}(\zeta_t)} + \underbrace{\left(2m^2 - m\right)}_{\text{cov}(\xi_{it},\xi_{js})} + \underbrace{(m-1)}_{\gamma_i} \\
&= (2m + 2m - m + m) + m \cdot n + n + 2m^2 - 1 \\
&= 2m^2 + 4m + mn + n - 1
\end{aligned}$$

Thus, the degrees of freedom for the ISGM are given by

$$\begin{aligned} df &= \#\text{known} - \#\text{estimated} \\ &= \frac{(mn)^2 + 3mn}{2} - \left(2m^2 + 4m + mn + n - 1\right) \\ &= \frac{(mn)^2 + 3mn}{2} - \left(\frac{4m^2 + 8m + 2mn + 2n - 2}{2}\right) \\ &= \frac{m^2n^2 - 4m^2 + mn - 8m - 2n + 2}{2}. \end{aligned}$$

Note that for models with autoregressive parameters (β_t) as well as partial or complete non-invariance of measurement parameters ($\alpha_{it}, \lambda_{it}, \gamma_{it}$), models with fewer df would result.

Appendix E
Model Identification with Two Indicators per Time Point

To demonstrate the limitations of a growth model with only two indicators per time point, it is sufficient to consider only the case with three time-points. That is, $i = \{1, 2\}$, $t = \{1, 2, 3\}$. The covariance matrix for this case, in terms of Y_{it} for all i, t is shown in Table E1.

Table E1
Covariance Matrix for 2 Indicators and 3 Time-points

		$i=1$ $t=1$	$i=2$ $t=1$	$i=1$ $t=2$	$i=2$ $t=2$	$i=1$ $t=3$	$i=2$ $t=3$
$j=1$	$s=1$	$\text{var}(Y_{11})$					
$j=2$	$s=1$	$\text{cov}(Y_{11}, Y_{21})$	$\text{var}(Y_{21})$				
$j=1$	$s=2$	$\text{cov}(Y_{11}, Y_{12})$	$\text{cov}(Y_{21}, Y_{12})$	$\text{var}(Y_{12})$			
$j=2$	$s=2$	$\text{cov}(Y_{11}, Y_{22})$	$\text{cov}(Y_{21}, Y_{22})$	$\text{cov}(Y_{12}, Y_{22})$	$\text{var}(Y_{22})$		
$j=1$	$s=3$	$\text{cov}(Y_{11}, Y_{13})$	$\text{cov}(Y_{21}, Y_{13})$	$\text{cov}(Y_{12}, Y_{13})$	$\text{cov}(Y_{22}, Y_{13})$	$\text{var}(Y_{13})$	
$j=2$	$s=3$	$\text{cov}(Y_{11}, Y_{23})$	$\text{cov}(Y_{21}, Y_{23})$	$\text{cov}(Y_{12}, Y_{23})$	$\text{cov}(Y_{22}, Y_{23})$	$\text{cov}(Y_{13}, Y_{23})$	$\text{var}(Y_{23})$

To show the general case, we first write out the complete model for the GSGM. This is accomplished by starting with the generic covariance relation for the autoregressive GSGM given in Table B1. The substitution of values for the indices into the general covariance equation is very straightforward.

However, as noted in Appendix B, $\beta_t \zeta_{t-1} + \delta_t$ must be recursively substituted for ζ_t (if $t > 1$ or $s > 1$). Then, the resulting covariance relation must be expanded and simplified. We demonstrate this for all except the trivial instance where $t = s = 1$. Note that the order in which the values (t, s) appear is not important $(\text{cov}(\zeta_t, \zeta_s) = \text{cov}(\zeta_s, \zeta_t))$.

$(1, 2) = (2, 1)$

$$\begin{aligned}
\text{cov}(\zeta_2, \zeta_1) &= \text{cov}(\beta_2\zeta_1 + \delta_2, \zeta_1) \\
&= \text{cov}(\beta_2\zeta_1, \zeta_1) + \text{cov}(\delta_2, \zeta_1) \\
&= \beta_2 \,\text{var}(\zeta_1)
\end{aligned}$$

$(1, 3) = (3, 1)$

$$\begin{aligned}
\text{cov}(\zeta_1,\ \zeta_3) &= \text{cov}(\zeta_1,\ \beta_3 [\beta_2\zeta_1 + \delta_2] + \delta_3) \\
&= \text{cov}(\zeta_1,\ \beta_3\beta_2\zeta_1 + \beta_3\delta_2 + \delta_3) \\
&= \beta_3\beta_2 \,\text{var}(\zeta_1)
\end{aligned}$$

$(2, 2)$

$$\begin{aligned}
\text{cov}(\zeta_2, \zeta_2) &= \text{cov}(\beta_2\zeta_1 + \delta_2, \beta_2\zeta_1 + \delta_2) \\
&= \text{cov}(\beta_2\zeta_1 + \delta_2, \beta_2\zeta_1) + \text{cov}(\beta_2\zeta_1 + \delta_t, \delta_2) \\
&= \text{cov}(\beta_2\zeta_1, \beta_2\zeta_1) + \underbrace{\text{cov}(\delta_2, \beta_2\zeta_1)}_{0} + \\
&\quad \underbrace{\text{cov}(\beta_2\zeta_1, \delta_2)}_{0} + \text{cov}(\delta_2, \delta_2) \\
&= \beta_2^2 \,\text{var}(\zeta_1) + \text{var}(\delta_2)
\end{aligned}$$

$(2,3)=(3,2)$

$$\begin{aligned}
\text{cov}(\zeta_2,\,\zeta_3) &= \text{cov}(\beta_2\zeta_1+\delta_2,\,\beta_3[\beta_2\zeta_1+\delta_2]+\delta_3)\\
&= \text{cov}(\beta_2\zeta_1,\,\beta_3[\beta_2\zeta_1+\delta_2]+\delta_3)+\\
&\quad\ \text{cov}(\delta_2,\,\beta_3[\beta_{21}\zeta_1+\delta_2]+\delta_3)\\
&= \beta_3\beta_2^2\,\text{cov}(\zeta_1,\,\zeta_1)+\text{cov}(\delta_2,\,\beta_3\beta_2\zeta_1+\beta_3\delta_2)+\\
&= \beta_3\beta_2^2\,\text{var}(\zeta_1)+\beta_3\,\text{var}(\delta_2,\delta_2)
\end{aligned}$$

$(3,3)$

$$\begin{aligned}
\text{cov}(\zeta_3,\,\zeta_3) &= \text{cov}(\beta_3[\beta_2\zeta_1+\delta_2]+\delta_3,\,\beta_3[\beta_2\zeta_1+\delta_2]+\delta_3)\\
&= \text{cov}(\beta_3[\beta_2\zeta_1+\delta_2],\,\beta_3[\beta_2\zeta_1+\delta_2])+\\
&\quad\ \underbrace{2\,\text{cov}(\beta_3[\beta_2\zeta_1+\delta_2],\,\delta_3)}_{0}+\\
&\quad\ \text{cov}(\delta_3,\,\delta_3)\\
&= \text{cov}(\beta_3\beta_2\zeta_1+\beta_3\delta_2,\,\beta_3\beta_2\zeta_1+\beta_3\delta_2)+\\
&\quad\ \text{cov}(\beta_3\beta_2\zeta_1,\,\beta_3\beta_2\zeta_1)+\\
&\quad\ \underbrace{2\,\text{cov}(\beta_3\beta_2\zeta_1,\,\beta_3\delta_2)}_{0}+\\
&\quad\ \text{cov}(\beta_3\delta_2,\,\beta_3\delta_2)+\\
&\quad\ \text{var}(\delta_3)\\
&= \beta_3^2\beta_2^2\,\text{var}(\zeta_1)+\beta_3^2\,\text{var}(\delta_2)+\text{var}(\delta_3)
\end{aligned}$$

The results from substituting all possible combinations for the specific values of i and t into this covariance equation, and simplifying these according to the model restrictions given in Equations (B.7)-(B.9) are shown in Table E3. Similarly, the results of this substitution for the case of uncorrelated state residual variables, simplified according to the restrictions given in Equations (B.2)-(B.4), are shown in Table E3.

We assume MI and uncorrelated state residual variables in order to examine when the model is not identified. The identification problem comes from the state residual (ζ_t) parameters. This is evident from the fact that the latent state residual variable for the first time point, ζ_1, appears in exactly three equations from Table E3:

$$\begin{aligned}
\text{var}(Y_{11}) &= \text{var}(\xi_{\text{int}})+\text{var}(\zeta_1)+\text{var}(\varepsilon_{11})\\
\text{cov}(Y_{11},Y_{21}) &= \lambda_{21}\,\text{var}(\xi_{\text{int}})+\gamma_{21}\,\text{var}(\zeta_1)\\
\text{var}(Y_{21}) &= \lambda_{21}^2\,\text{var}(\xi_{\text{int}})+\gamma_{21}^2\,\text{var}(\zeta_1)+\text{var}(\varepsilon_{21})
\end{aligned}$$

These are also the only equations in which the parameter γ_{21} appears. In these equations, $\text{var}(Y_{11})$, $\text{var}(Y_{21})$, and $\text{cov}(Y_{11},Y_{21})$ are known, while $\text{var}(\xi_{\text{int}})$ and λ_{21} can be identified from the other model equations. The parameters γ_{21}, $\text{var}(\zeta_1)$, $\text{var}(\varepsilon_{11})$ and $\text{var}(\varepsilon_{21})$ are not known. These parameters do not appear elsewhere and thus cannot be estimated from other information. Without formal proof, we note that the situation remains unchanged regardless of the number of time-points present. This leads to the conclusion that

Table E2

Variance and covariance equations for the GSGM-AR with two indicators and three time points.

$\text{var}(Y_{11})$	$=$	$\text{var}(\xi_{\text{int}}) + \gamma_{11}^2 \, \text{var}(\zeta_1) + \text{var}(\varepsilon_{11})$
$\text{cov}(Y_{11}, Y_{21})$	$=$	$\lambda_{21} \, \text{var}(\xi_{\text{int}}) + \gamma_{21} \, \text{var}(\zeta_1)$
$\text{cov}(Y_{11}, Y_{12})$	$=$	$\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}}) + \beta_2 \, \text{var}(\zeta_1)$
$\text{cov}(Y_{11}, Y_{22})$	$=$	$\lambda_{22} [\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] + \gamma_{22}\beta_2 \, \text{var}(\zeta_1)$
$\text{cov}(Y_{11}, Y_{13})$	$=$	$[2 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] + \beta_3\beta_2 \, \text{var}(\zeta_1)$
$\text{cov}(Y_{11}, Y_{23})$	$=$	$\lambda_{23} [2 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] + \gamma_{23}\beta_3\beta_2 \, \text{var}(\zeta_1)$
$\text{var}(Y_{21})$	$=$	$\lambda_{21}^2 \, \text{var}(\xi_{\text{int}}) + \lambda_{21}^2 \, \text{var}(\zeta_1) + \text{var}(\varepsilon_{21})$
$\text{cov}(Y_{21}, Y_{12})$	$=$	$\lambda_{21} [\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] + \gamma_{21}\beta_2 \, \text{var}(\zeta_1)$
$\text{cov}(Y_{21}, Y_{22})$	$=$	$\lambda_{21}\lambda_{22} [\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] + \gamma_{21}\gamma_{22}\beta_2 \, \text{var}(\zeta_1)$
$\text{cov}(Y_{21}, Y_{13})$	$=$	$\lambda_{21} [2 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] + \gamma_{21}\beta_3\beta_2 \, \text{var}(\zeta_1)$
$\text{cov}(Y_{21}, Y_{23})$	$=$	$\lambda_{21}\lambda_{23} [2 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] + \gamma_{21}\gamma_{23}\beta_3\beta_2 \, \text{var}(\zeta_1)$
$\text{var}(Y_{12})$	$=$	$2 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}}) +$
		$\beta_2^2 \, \text{var}(\zeta_1) + \text{var}(\delta_2) + \text{var}(\varepsilon_{12})$
$\text{cov}(Y_{12}, Y_{22})$	$=$	$\lambda_{22} [2 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] +$
		$\gamma_{22} \left[\beta_2^2 \, \text{var}(\zeta_1) + \text{var}(\delta_2)\right]$
$\text{cov}(Y_{12}, Y_{13})$	$=$	$[3 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 2 \, \text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] +$
		$\left[\beta_3\beta_2^2 \, \text{var}(\zeta_1) + \beta_3 \, \text{var}(\delta_2)\right]$
$\text{cov}(Y_{12}, Y_{23})$	$=$	$\lambda_{23} [3 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 2 \, \text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] +$
		$\gamma_{23} \left[\beta_3\beta_2^2 \, \text{var}(\zeta_1) + \beta_3 \, \text{var}(\delta_2)\right]$
$\text{var}(Y_{22})$	$=$	$\lambda_{22}^2 [2 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] +$
		$\gamma_{22}^2 \left[\beta_2^2 \, \text{var}(\zeta_1) + \text{var}(\delta_2)\right] + \text{var}(\varepsilon_{22})$
$\text{cov}(Y_{22}, Y_{13})$	$=$	$\lambda_{22} [3 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 2 \, \text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] +$
		$\gamma_{22} \left[\beta_3\beta_2^2 \, \text{var}(\zeta_1) + \beta_3 \, \text{var}(\delta_2)\right]$
$\text{cov}(Y_{22}, Y_{23})$	$=$	$\lambda_{22}\lambda_{23} [3 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 2 \, \text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] +$
		$\gamma_{22}\gamma_{23} \left[\beta_3\beta_2^2 \, \text{var}(\zeta_1) + \beta_3 \, \text{var}(\delta_2)\right]$
$\text{var}(Y_{13})$	$=$	$4 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 4 \, \text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}}) +$
		$\beta_3^2\beta_2^2 \, \text{var}(\zeta_1) + \beta_3^2 \, \text{var}(\delta_2) + \text{var}(\delta_3) + \text{var}(\varepsilon_{13})$
$\text{cov}(Y_{13}, Y_{23})$	$=$	$\lambda_{23} [4 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 4 \, \text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] +$
		$\gamma_{23} \left[\beta_3^2\beta_2^2 \, \text{var}(\zeta_1) + \beta_3^2 \, \text{var}(\delta_2) + \text{var}(\delta_3)\right]$
$\text{var}(Y_{23})$	$=$	$\lambda_{23}^2 [4 \, \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 4 \, \text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})] +$
		$\gamma_{23}^2 \left[\beta_3^2\beta_2^2 \, \text{var}(\zeta_1) + \beta_3^2 \, \text{var}(\delta_2) + \text{var}(\delta_3)\right] + \text{var}(\varepsilon_{23})$

Table E3

Variance and covariance equations for the GSGM with two indicators and three time points.

$$
\begin{aligned}
\text{var}(Y_{11}) &= \text{var}(\xi_{\text{int}}) + \text{var}(\zeta_1) + \text{var}(\varepsilon_{11}) \\
\text{cov}(Y_{11}, Y_{21}) &= \lambda_{21}\,\text{var}(\xi_{\text{int}}) + \gamma_{21}\,\text{var}(\zeta_1) \\
\text{cov}(Y_{11}, Y_{12}) &= \text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}}) \\
\text{cov}(Y_{11}, Y_{22}) &= \lambda_{22}\left[\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] \\
\text{cov}(Y_{11}, Y_{13}) &= 2\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}}) \\
\text{cov}(Y_{11}, Y_{23}) &= \lambda_{23}\left[2\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] \\
\text{var}(Y_{21}) &= \lambda_{21}^2\,\text{var}(\xi_{\text{int}}) + \gamma_{21}^2\,\text{var}(\zeta_1) + \text{var}(\varepsilon_{21}) \\
\text{cov}(Y_{21}, Y_{12}) &= \lambda_{21}\left[\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] \\
\text{cov}(Y_{21}, Y_{22}) &= \lambda_{21}\lambda_{22}\left[\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] \\
\text{cov}(Y_{21}, Y_{13}) &= \lambda_{21}\left[2\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] \\
\text{cov}(Y_{21}, Y_{23}) &= \lambda_{21}\lambda_{23}\left[2\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] \\
\text{var}(Y_{12}) &= \left[2\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] + \\
&\quad \text{var}(\zeta_2) + \text{var}(\varepsilon_{12}) \\
\text{cov}(Y_{12}, Y_{22}) &= \lambda_{22}\left[2\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] + \\
&\quad \gamma_{22}\,\text{var}(\zeta_2) \\
\text{cov}(Y_{12}, Y_{13}) &= 3\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 2\,\text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}}) \\
\text{cov}(Y_{12}, Y_{23}) &= \lambda_{23}\left[3\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 2\,\text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] \\
\text{var}(Y_{22}) &= \lambda_{22}^2\left[2\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + \text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] + \\
&\quad \gamma_{22}^2\,\text{var}(\zeta_2) + \text{var}(\varepsilon_{22}) \\
\text{cov}(Y_{22}, Y_{13}) &= \lambda_{22}\left[3\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 2\,\text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] \\
\text{cov}(Y_{22}, Y_{23}) &= \lambda_{22}\lambda_{23}\left[3\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 2\,\text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] \\
\text{var}(Y_{13}) &= 4\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 4\,\text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}}) + \\
&\quad \text{var}(\zeta_3) + \text{var}(\varepsilon_{13}) \\
\text{cov}(Y_{13}, Y_{23}) &= \lambda_{23}\left[4\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 4\,\text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] + \\
&\quad \gamma_{23}\,\text{var}(\zeta_3) \\
\text{var}(Y_{23}) &= \lambda_{23}^2\left[4\,\text{cov}(\xi_{\text{int}}, \xi_{\text{lin}}) + 4\,\text{var}(\xi_{\text{lin}}) + \text{var}(\xi_{\text{int}})\right] + \\
&\quad \gamma_{23}^2\,\text{var}(\zeta_3) + \text{var}(\varepsilon_{23})
\end{aligned}
$$

the GSGM is not generally identified with only two indicators at each time-point, unless further constraints are imposed or effects are added to the model. If the parameters γ_{2t} are known, the remaining unknown parameters in these equations can be estimated. This can be accomplished by fixing the state residual loadings γ_{2t} to some value (i.e., $\gamma_{1t} = \gamma_{2t} = 1$). Alternatively, the SGM, which we have shown to be a special case of the GSGM in which $\gamma_{it} = \lambda_{it}$ can be used to adequately constrain this model, since the λ_{it} parameters can be estimated from other model information.

Another case in which the model information is sufficient for the parameters γ_{2t} and $\text{var}(\zeta_t)$ to be estimated occurs when there is a significant autoregressive structure. The γ_{2t} loadings in this case are related by the β_t parameters. This can be seen by comparing the covariance relations in Table E2 to those in Table E3, paying particular attention to the parameters containing the terms containing ζ_1 and δ_t.

Appendix F
Variance Components and Coefficients

In each of the three growth models, the observed variance can be partitioned into variance due to the growth process, variance due to state residual variability, and measurement error variance as shown in Figure F1.

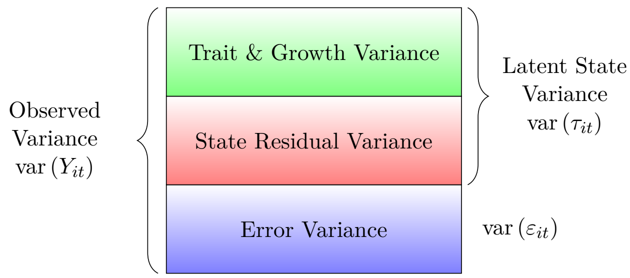


Figure F1. Variance Decomposition in the SGM, GSGM, and ISGM

Based on the variance decomposition in each model (see Table F1), several coefficients can be defined (Eid, Courvoisier, & Lischetzke, 2012):

Coefficient of reliability:

$$\text{Rel}\left(Y_{it}\right) = \frac{\text{State Variance}}{\text{Observed Variance}} = \frac{\text{var}\left(\tau_{it}\right)}{\text{var}\left(Y_{it}\right)}.$$

Coefficient of consistency:

$$\text{Con}\left(\tau_{it}\right) = \frac{\text{Trait \& Growth Variance}}{\text{State Variance}}.$$

Coefficient of occasion-specificity:

$$\text{OSpec}\left(\tau_{it}\right) = \frac{\text{State Residual Variance}}{\text{State Variance}}.$$

Table F1
Variance Decomposition

Model	Variance Component	Equation
SGM		
	Trait & Growth Variance	$\lambda_{it}^2 \,\text{var}\left(\xi_{\text{int}}\right) + (t-1)^2 \lambda_{it}^2 \,\text{var}\left(\xi_{\text{lin}}\right) + 2\,(t-1)\,\lambda_{it}^2 \,\text{cov}\left(\xi_{\text{int}},\, \xi_{\text{lin}}\right)$
	State Residual Variance	$\lambda_{it}^2 \,\text{var}\left(\zeta_{t-1}\right)$
	Error Variance	$\text{var}\left(\varepsilon_{it}\right)$
GSGM		
	Trait & Growth Variance	$\lambda_{it}^2 \,\text{var}\left(\xi_{\text{int}}\right) + (t-1)^2 \lambda_{it}^2 \,\text{var}\left(\xi_{\text{lin}}\right) + 2\,(t-1)\,\lambda_{it}^2 \,\text{cov}\left(\xi_{\text{int}},\, \xi_{\text{lin}}\right)$
	State Residual Variance	$\gamma_{it}^2 \,\text{var}\left(\zeta_t\right)$
	Error Variance	$\text{var}\left(\varepsilon_{it}\right)$
ISGM		
	Trait & Growth Variance	$\text{var}\left(\xi_{\text{int}_i}\right) + (t-1)^2 \,\text{var}\left(\xi_{\text{lin}_i}\right) + 2\,(t-1)\,\text{cov}\left(\xi_{\text{int}_i},\, \xi_{\text{lin}_i}\right)$
	State Residual Variance	$\gamma_{it}^2 \,\text{var}\left(\zeta_t\right)$
	Error Variance	$\text{var}\left(\varepsilon_{it}\right)$

Note that whereas the reliability coefficient is defined for the observed variables Y_{it}, the Con and OSpec coefficients are defined for the true score variables τ_{it}, so as to partition the total systematic variance into growth versus state residual variance. Below we show the specific calculations for each model.

SGM

Total Variance:

$$\mathrm{var}\left(Y_{it}\right) = \lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{int}}\right) + (t-1)^2\,\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{lin}}\right) + 2\,(t-1)\,\lambda_{it}^2\,\mathrm{cov}\left(\xi_{\mathrm{int}},\,\xi_{\mathrm{lin}}\right) + \lambda_{it}^2\,\mathrm{var}\left(\zeta_t\right) + \mathrm{var}\left(\varepsilon_{it}\right).$$

Reliability Coefficient:

$$\mathrm{Rel}\left(Y_{it}\right) = \frac{\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{int}}\right) + (t-1)^2\,\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{lin}}\right) + 2\,(t-1)\,\lambda_{it}^2\,\mathrm{cov}\left(\xi_{\mathrm{int}},\,\xi_{\mathrm{lin}}\right) + \lambda_{it}^2\,\mathrm{var}\left(\zeta_t\right)}{\mathrm{var}\left(Y_{it}\right)}.$$

Consistency Coefficient:

$$\begin{aligned}\mathrm{Con}\left(\tau_t\right) &= \frac{\mathrm{var}\left(\xi_{\mathrm{int}}\right) + (t-1)^2\,\mathrm{var}\left(\xi_{\mathrm{lin}}\right) + 2\,(t-1)\,\mathrm{cov}\left(\xi_{\mathrm{int}},\,\xi_{\mathrm{lin}}\right)}{\mathrm{var}\left(\tau_t\right)}, \\ &= \frac{\mathrm{var}\left(\xi_{\mathrm{int}}\right) + (t-1)^2\,\mathrm{var}\left(\xi_{\mathrm{lin}}\right) + 2\,(t-1)\,\mathrm{cov}\left(\xi_{\mathrm{int}},\,\xi_{\mathrm{lin}}\right)}{\mathrm{var}\left(\xi_{\mathrm{int}}\right) + (t-1)^2\,\mathrm{var}\left(\xi_{\mathrm{lin}}\right) + 2\,(t-1)\,\mathrm{cov}\left(\xi_{\mathrm{int}},\,\xi_{\mathrm{lin}}\right) + \mathrm{var}\left(\zeta_t\right)}.\end{aligned}$$

Occasion-specificity Coefficient:

$$\begin{aligned}\mathrm{OSpec}\left(\tau_t\right) &= \frac{\mathrm{var}\left(\zeta_t\right)}{\mathrm{var}\left(\tau_t\right)}, \\ &= \frac{\mathrm{var}\left(\zeta_t\right)}{\mathrm{var}\left(\xi_{\mathrm{int}}\right) + (t-1)^2\,\mathrm{var}\left(\xi_{\mathrm{lin}}\right) + 2\,(t-1)\,\mathrm{cov}\left(\xi_{\mathrm{int}},\,\xi_{\mathrm{lin}}\right) + \mathrm{var}\left(\zeta_t\right)}.\end{aligned}$$

GSGM

Total Variance:

$$\mathrm{var}\left(Y_{it}\right) = \lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{int}}\right) + (t-1)^2\,\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{lin}}\right) + 2\,(t-1)\,\lambda_{it}^2\,\mathrm{cov}\left(\xi_{\mathrm{int}},\,\xi_{\mathrm{lin}}\right) + \gamma_{it}^2\,\mathrm{var}\left(\zeta_t\right) + \mathrm{var}\left(\varepsilon_{it}\right).$$

Reliability Coefficient:

$$\mathrm{Rel}\left(Y_{it}\right) = \frac{\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{int}}\right) + (t-1)^2\,\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{lin}}\right) + 2\,(t-1)\,\lambda_{it}^2\,\mathrm{cov}\left(\xi_{\mathrm{int}},\,\xi_{\mathrm{lin}}\right) + \gamma_{it}^2\,\mathrm{var}\left(\zeta_t\right)}{\mathrm{var}\left(Y_{it}\right)}.$$

Consistency Coefficient:

$$\begin{aligned}\mathrm{Con}\left(\tau_{it}\right) &= \frac{\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{int}}\right) + (t-1)^2\,\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{lin}}\right) + 2\,(t-1)\,\lambda_{it}^2\,\mathrm{cov}\left(\xi_{\mathrm{int}},\,\xi_{\mathrm{lin}}\right)}{\mathrm{var}\left(\tau_{it}\right)}, \\ &= \frac{\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{int}}\right) + (t-1)^2\,\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{lin}}\right) + 2\,(t-1)\,\lambda_{it}^2\,\mathrm{cov}\left(\xi_{\mathrm{int}},\,\xi_{\mathrm{lin}}\right)}{\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{int}}\right) + (t-1)^2\,\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{lin}}\right) + 2\,(t-1)\,\lambda_{it}^2\,\mathrm{cov}\left(\xi_{\mathrm{int}},\,\xi_{\mathrm{lin}}\right) + \gamma_{it}^2\,\mathrm{var}\left(\zeta_t\right)}.\end{aligned}$$

Occasion-specificity Coefficient:

$$\begin{aligned}\mathrm{OSpec}\left(\tau_t\right) &= \frac{\gamma_{it}^2\,\mathrm{var}\left(\zeta_t\right)}{\mathrm{var}\left(\tau_{it}\right)}, \\ &= \frac{\gamma_{it}^2\,\mathrm{var}\left(\zeta_t\right)}{\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{int}}\right) + (t-1)^2\,\lambda_{it}^2\,\mathrm{var}\left(\xi_{\mathrm{lin}}\right) + 2\,(t-1)\,\lambda_{it}^2\,\mathrm{cov}\left(\xi_{\mathrm{int}},\,\xi_{\mathrm{lin}}\right) + \gamma_{it}^2\,\mathrm{var}\left(\zeta_t\right)}.\end{aligned}$$

ISGM

Total Variance:

$$\text{var}(Y_{it}) = \text{var}(\xi_{\text{int}_i}) + (t-1)^2\,\text{var}(\xi_{\text{lin}_i}) + 2(t-1)\,\text{cov}(\xi_{\text{int}_i}, \xi_{\text{lin}_i}) + \gamma_{it}^2\,\text{var}(\zeta_t) + \text{var}(\varepsilon_{it})\,.$$

Reliability Coefficient:

$$\text{Rel}(Y_{it}) = \frac{\text{var}(\xi_{\text{int}_i}) + (t-1)^2\,\text{var}(\xi_{\text{lin}_i}) + 2(t-1)\,\text{cov}(\xi_{\text{int}_i}, \xi_{\text{lin}_i}) + \gamma_{it}^2\,\text{var}(\zeta_t)}{\text{var}(Y_{it})}.$$

Consistency Coefficient:

$$\begin{aligned}\text{Con}(\tau_{it}) &= \frac{\text{var}(\xi_{\text{int}_i}) + (t-1)^2\,\text{var}(\xi_{\text{lin}_i}) + 2(t-1)\,\text{cov}(\xi_{\text{int}_i}, \xi_{\text{lin}_i})}{\text{var}(\tau_{it})},\\ &= \frac{\text{var}(\xi_{\text{int}_i}) + (t-1)^2\,\text{var}(\xi_{\text{lin}_i}) + 2(t-1)\,\text{cov}(\xi_{\text{int}_i}, \xi_{\text{lin}_i})}{\text{var}(\xi_{\text{int}_i}) + (t-1)^2\,\text{var}(\xi_{\text{lin}_i}) + 2(t-1)\,\text{cov}(\xi_{\text{int}_i}, \xi_{\text{lin}_i}) + \gamma_{it}^2\,\text{var}(\zeta_t)}.\end{aligned}$$

Occasion-specificity Coefficient:

$$\text{OSpec}(\tau_{it}) = \frac{\gamma_{it}^2\,\text{var}(\zeta_t)}{\text{var}(\xi_{\text{int}_i}) + (t-1)^2\,\text{var}(\xi_{\text{lin}_i}) + 2(t-1)\,\text{cov}(\xi_{\text{int}_i}, \xi_{\text{lin}_i}) + \gamma_{it}^2\,\text{var}(\zeta_t)}.$$

Note that for the SGM, the consistency coefficient and its complement (OSpec) are dependent only upon τ_t. Thus, this coefficient is the same for all indicators measured at the same time point in this model. However, for both the GSGM and ISGM, Con and OSpec are indicator-specific (dependent on τ_{it}) because there is not a common latent state factor for each time-point in these two models. Additional coefficients can be defined in the case of an autoregressive structure among the state residuals as shown in Eid et al. (2012).

Appendix G
Mplus Model Inputs for SGM, GSGM, and ISGM

The order and names used for the following model specifications match the specification in Table 3 (of the main paper). The data file used for this analysis is provided for practice purposes only. It is available for download at `http://supp.apa.org/psycarticles/supplemental/met0000018/met0000018_supp.html`. Publication of this data in any form, including results from analyzing this data is expressly forbidden. For requests to use this data for publication or other purposes, please contact Dr. David Cole at david.cole@vanderbilt.edu.

Model 1a: SGM

filename: 1a_SGM_LinearGrowth_AlphaLamInv.inp

```
TITLE: Second-Order Multiple-Indicator Growth Model.
       Linear Growth.
       First-order representation.
          (after Schmid-Leiman Transformation).
DATA: file = Anxiety_Data_3_Indicators_4_Waves.dat;
VARIABLE:
names = grpID
        y11 y21 y31
        y12 y22 y32
        y13 y23 y33
        y14 y24 y34;
usevariables = y11 y21 y31
               y12 y22 y32
               y13 y23 y33
               y14 y24 y34;
missing = all(-99);
MODEL:
! Common xi. This is the trait intercept.
! Set the first trait loading (lambda) to 1.
! Estimate the other two.
! Measurement Invariance on lambda not assumed.
xi_int by y11@1 (lambda11)
          y21   (lambda21)
          y31   (lambda31)
          y12@1 (lambda12)
          y22   (lambda22)
          y32   (lambda32)
          y13@1 (lambda13)
          y23   (lambda23)
          y33   (lambda33)
          y14@1 (lambda14)
          y24   (lambda24)
```

```
          y34   (lambda34);
! Estimate the mean of the trait factor.
! (zero by default in Mplus).
[xi_int*];
! Variance of the trait factor (xi_int).
xi_int (xi_intv);
! Growth Model. This is the trait slope.
! Wave 1 is missing, because lambda would be multiplied by 0.
! Multiply by 1*lambda for the second wave, by 2*lambda for
! the third, and 3*lambda for the fourth.
xi_lin by y12@1 (lambda12)
          y22   (lambda22)
          y32   (lambda32)
          y13@2 (lambda13d)
          y23   (lambda23d)
          y33   (lambda33d)
          y14@3 (lambda14t)
          y24   (lambda24t)
          y34   (lambda34t);
! Estimate the mean of the growth factor.
! (zero by default in Mplus).
[xi_lin*];
! Variance of the growth factor (xi_lin).
xi_lin (xi_linv);
! These loadings must be the same as the trait
! loadings given above...a constraint of the SGM.
zeta1 by y11@1 (lambda11)
         y21   (lambda21)
         y31   (lambda31);
zeta2 by y12@1 (lambda12)
         y22   (lambda22)
         y32   (lambda32);
zeta3 by y13@1 (lambda13)
         y23   (lambda23)
         y33   (lambda33);
zeta4 by y14@1 (lambda14)
         y24   (lambda24)
         y34   (lambda34);
! Set intercepts of state residual factors to zero.
! (would otherwise be estimated as the default in Mplus)
[zeta1-zeta4@0];

! Delta error variances on each zeta.
zeta1 (zeta1var);
zeta2 (zeta2var);
```

```
zeta3 (zeta3var);
zeta4 (zeta4var);
! Constant intercepts (alpha_it).
! Set the intercept on the first indicator
! (alpha1) to 0, and estimate the other
! two (alpha2, alpha3).
! This assumes alpha-MI.
[y11@0]; [y21*] (alpha21); [y31*] (alpha31);
[y12@0]; [y22*] (alpha22); [y32*] (alpha32);
[y13@0]; [y23*] (alpha23); [y33*] (alpha33);
[y14@0]; [y24*] (alpha24); [y34*] (alpha34);
! Estimate the Error Variances.
y11 (ev11); y21 (ev21); y31 (ev31);
y12 (ev12); y22 (ev22); y32 (ev32);
y13 (ev13); y23 (ev23); y33 (ev33);
y14 (ev14); y24 (ev24); y34 (ev34);
! Non-admissible correlations.
xi_int with zeta1-zeta4@0;
xi_lin with zeta1-zeta4@0;
zeta1 with zeta2-zeta4@0;
zeta2 with zeta3-zeta4@0;
zeta3 with zeta4@0;
MODEL CONSTRAINT:
! lambdaid is double lambda i.
lambda23d = lambda23*2;
lambda33d = lambda33*2;
! lambdait is triple lambda i.
lambda24t = lambda24*3;
lambda34t = lambda34*3;
OUTPUT: sampstat stdyx;
```

Model 1b: SGM, α, λ-MI

filename: 1b_SGM_LinearGrowth_AlphaLamInv.inp

```
TITLE: Second-Order Multiple-Indicator Growth Model.
       Measurement invariance assumed on alpha and lambda.
       Linear Growth.
       First-order representation.
          (after Schmid-Leiman Transformation).
DATA: file = Anxiety_Data_3_Indicators_4_Waves.dat;
VARIABLE:
names = grpID
        y11 y21 y31
        y12 y22 y32
        y13 y23 y33
```

```
        y14 y24 y34;
usevariables = y11 y21 y31
               y12 y22 y32
               y13 y23 y33
               y14 y24 y34;
missing = all(-99);
MODEL:
! Common xi. This is the trait intercept.
! Set the first trait loading (lambda) to 1.
! Estimate the other two.
! This assumes Measurement Invariance on lambda.
! Do this if the lambdas have to be the same
! for a given test across testing occasions.
xi_int by y11@1 y21 y31 (lambda1-lambda3)
          y12@1 y22 y32 (lambda1-lambda3)
          y13@1 y23 y33 (lambda1-lambda3)
          y14@1 y24 y34 (lambda1-lambda3);
! Estimate the mean of the trait factor.
! (zero by default in Mplus).
[xi_int*];
! Variance of the trait factor (xi_int).
xi_int (xi_intv);
! Growth Model. This is the trait slope.
! Wave 1 is missing, because lambda would be multiplied by 0.
! Multiply by 1*lambda for the second wave, by 2*lambda for
! the third, and 3*lambda for the fourth.
xi_lin by y12@1 (lambda1)
          y22   (lambda2)
          y32   (lambda3)
          y13@2 (lambda1d)
          y23   (lambda2d)
          y33   (lambda3d)
          y14@3 (lambda1t)
          y24   (lambda2t)
          y34   (lambda3t);
! Estimate the mean of the growth factor.
! (zero by default in Mplus).
[xi_lin*];
! Variance of the growth factor (xi_lin).
xi_lin (xi_linv);
! These loadings must be the same as the trait
! loadings given above...a constraint of the SGM.
zeta1 by y11@1 y21 y31 (lambda1-lambda3);
zeta2 by y12@1 y22 y32 (lambda1-lambda3);
zeta3 by y13@1 y23 y33 (lambda1-lambda3);
```

```
zeta4 by y14@1 y24 y34 (lambda1-lambda3);
! Set intercepts of state residual factors to zero.
! (would otherwise be estimated as the default in Mplus)
[zeta1-zeta4@0];

! Delta error variances on each zeta.
zeta1 (zeta1var);
zeta2 (zeta2var);
zeta3 (zeta3var);
zeta4 (zeta4var);
! Constant intercepts (alpha_it).
! Set the intercept on the first indicator
! (alpha1) to 0, and estimate the other
! two (alpha2, alpha3).
! This assumes alpha-MI.
[y11@0]; [y21*] (alpha2); [y31*] (alpha3);
[y12@0]; [y22*] (alpha2); [y32*] (alpha3);
[y13@0]; [y23*] (alpha2); [y33*] (alpha3);
[y14@0]; [y24*] (alpha2); [y34*] (alpha3);
! Estimate the Error Variances.
y11 (ev11); y21 (ev21); y31 (ev31);
y12 (ev12); y22 (ev22); y32 (ev32);
y13 (ev13); y23 (ev23); y33 (ev33);
y14 (ev14); y24 (ev24); y34 (ev34);
! Non-admissible correlations.
xi_int with zeta1-zeta4@0;
xi_lin with zeta1-zeta4@0;
zeta1 with zeta2-zeta4@0;
zeta2 with zeta3-zeta4@0;
zeta3 with zeta4@0;
MODEL CONSTRAINT:
! lambdaid is double lambda i.
lambda2d = lambda2*2;
lambda3d = lambda3*2;
! lambdait is triple lambda i.
lambda2t = lambda2*3;
lambda3t = lambda3*3;
OUTPUT: sampstat stdyx;
```

Model 2a: GSGM

filename: 2a_GSGM_LinearGrowth.inp

```
TITLE: Generalized Second-Order Multiple-Indicator Growth Model.
       Linear Growth.
       First-order model representation.
```

```
          (after Schmid-Leiman Transformation).
DATA: file = Anxiety_Data_3_Indicators_4_Waves.dat;
VARIABLE:
names = grpID
        y11 y21 y31
        y12 y22 y32
        y13 y23 y33
        y14 y24 y34;
usevariables = y11 y21 y31
               y12 y22 y32
               y13 y23 y33
               y14 y24 y34;
missing = all(-99);
MODEL:
! Common xi. This is the trait intercept.
! Set the first trait loading (lambda) to 1.
! Estimate the other two.
! Measurement Invariance on lambda not assumed.
xi_int by y11@1 (lambda11)
          y21   (lambda21)
          y31   (lambda31)
          y12@1 (lambda12)
          y22   (lambda22)
          y32   (lambda32)
          y13@1 (lambda13)
          y23   (lambda23)
          y33   (lambda33)
          y14@1 (lambda14)
          y24   (lambda24)
          y34   (lambda34);
! Estimate the mean of the trait factor.
! (zero by default in Mplus).
[xi_int*];
! Variance of the trait factor (xi_int).
xi_int (xi_intv);
! Growth Model. This is the trait slope.
! Wave 1 is missing, because lambda would be multiplied by 0.
! Multiply by 1*lambda for the second wave, by 2*lambda for
! the third, and 3*lambda for the fourth.
xi_lin by y12@1 (lambda12)
          y22   (lambda22)
          y32   (lambda32)
          y13@2 (lambda13d)
          y23   (lambda23d)
          y33   (lambda33d)
```

```
          y14@3 (lambda14t)
          y24   (lambda24t)
          y34   (lambda34t);
! Estimate the mean of the growth factor.
! (zero by default in Mplus).
[xi_lin*];
! Variance of the growth factor (xi_lin).
xi_lin (xi_linv);
! These loadings are NOT the same as the trait
! loadings given above. This is the difference
! between the GSGM and the SGM.
! Do not assume measurement invariance of the state residual
! factor loadings (gamma_it=gamma_is).
zeta1 by y11@1 (gamma11)
         y21   (gamma21)
         y31   (gamma31);
zeta2 by y12@1 (gamma12)
         y22   (gamma22)
         y32   (gamma32);
zeta3 by y13@1 (gamma13)
         y23   (gamma23)
         y33   (gamma33);
zeta4 by y14@1 (gamma14)
         y24   (gamma24)
         y34   (gamma34);
! Set intercepts of state residual factors to zero.
! (would otherwise be estimated as the default in Mplus)
[zeta1-zeta4@0];

! Delta error variances on each zeta.
zeta1 (zeta1var);
zeta2 (zeta2var);
zeta3 (zeta3var);
zeta4 (zeta4var);
! Constant intercepts (alpha_it).
! Set the intercept on the first indicator
! (alpha1) to 0, and estimate the other
! two (alpha2, alpha3).
! This assumes alpha-MI.
[y11@0]; [y21*] (alpha21); [y31*] (alpha31);
[y12@0]; [y22*] (alpha22); [y32*] (alpha32);
[y13@0]; [y23*] (alpha23); [y33*] (alpha33);
[y14@0]; [y24*] (alpha24); [y34*] (alpha34);
! Estimate the Error Variances.
y11 (ev11); y21 (ev21); y31 (ev31);
```

```
y12 (ev12); y22 (ev22); y32 (ev32);
y13 (ev13); y23 (ev23); y33 (ev33);
y14 (ev14); y24 (ev24); y34 (ev34);
! Non-admissible correlations.
xi_int with zeta1-zeta4@0;
xi_lin with zeta1-zeta4@0;
zeta1 with zeta2-zeta4@0;
zeta2 with zeta3-zeta4@0;
zeta3 with zeta4@0;
MODEL CONSTRAINT:
! lambdaid is double lambda i.
lambda23d = lambda23*2;
lambda33d = lambda33*2;
! lambdait is triple lambda i.
lambda24t = lambda24*3;
lambda34t = lambda34*3;
OUTPUT: sampstat stdyx;
```

Model 2b: GSGM, α, λ-MI

filename: 2b__GSGM__LinearGrowth__AlphaLamInv.inp

```
TITLE: Generalized Second-Order Multiple-Indicator Growth Model.
       Measurement invariance assumed on alpha and lambda.
       Linear Growth.
       First-order model representation.
          (after Schmid-Leiman Transformation).
DATA: file = Anxiety_Data_3_Indicators_4_Waves.dat;
VARIABLE:
names = grpID
        y11 y21 y31
        y12 y22 y32
        y13 y23 y33
        y14 y24 y34;
usevariables = y11 y21 y31
               y12 y22 y32
               y13 y23 y33
               y14 y24 y34;
missing = all(-99);
MODEL:
! Common xi. This is the trait intercept.
! Set the first trait loading (lambda) to 1.
! Estimate the other two.
! This assumes Measurement Invariance on lambda.
! Do this if the lambdas have to be the same
! for a given test across testing occasions.
```

```
xi_int by y11@1 y21 y31 (lambda1-lambda3)
          y12@1 y22 y32 (lambda1-lambda3)
          y13@1 y23 y33 (lambda1-lambda3)
          y14@1 y24 y34 (lambda1-lambda3);
! Estimate the mean of the trait factor.
! (zero by default in Mplus).
[xi_int*];
! Variance of the trait factor (xi_int).
xi_int (xi_intv);
! Growth Model. This is the trait slope.
! Wave 1 is missing, because lambda would be multiplied by 0.
! Multiply by 1*lambda for the second wave, by 2*lambda for
! the third, and 3*lambda for the fourth.
xi_lin by y12@1 (lambda1)
          y22   (lambda2)
          y32   (lambda3)
          y13@2 (lambda1d)
          y23   (lambda2d)
          y33   (lambda3d)
          y14@3 (lambda1t)
          y24   (lambda2t)
          y34   (lambda3t);
! Estimate the mean of the growth factor.
! (zero by default in Mplus).
[xi_lin*];
! Variance of the growth factor (xi_lin).
xi_lin (xi_linv);
! These loadings are NOT the same as the trait
! loadings given above. This is the difference
! between the GSGM and the SGM.
! Do not assume measurement invariance of the state residual
! factor loadings (gamma_it=gamma_is).
zeta1 by y11@1 (gamma11)
         y21   (gamma21)
         y31   (gamma31);
zeta2 by y12@1 (gamma12)
         y22   (gamma22)
         y32   (gamma32);
zeta3 by y13@1 (gamma13)
         y23   (gamma23)
         y33   (gamma33);
zeta4 by y14@1 (gamma14)
         y24   (gamma24)
         y34   (gamma34);
! Set intercepts of state residual factors to zero.
```

```
! (would otherwise be estimated as the default in Mplus)
[zeta1-zeta4@0];

! Delta error variances on each zeta.
zeta1 (zeta1var);
zeta2 (zeta2var);
zeta3 (zeta3var);
zeta4 (zeta4var);
! Constant intercepts (alpha_it).
! Set the intercept on the first indicator
! (alpha1) to 0, and estimate the other
! two (alpha2, alpha3).
! This assumes alpha-MI.
[y11@0]; [y21*] (alpha2); [y31*] (alpha3);
[y12@0]; [y22*] (alpha2); [y32*] (alpha3);
[y13@0]; [y23*] (alpha2); [y33*] (alpha3);
[y14@0]; [y24*] (alpha2); [y34*] (alpha3);
! Estimate the Error Variances.
y11 (ev11); y21 (ev21); y31 (ev31);
y12 (ev12); y22 (ev22); y32 (ev32);
y13 (ev13); y23 (ev23); y33 (ev33);
y14 (ev14); y24 (ev24); y34 (ev34);
! Non-admissible correlations.
xi_int with zeta1-zeta4@0;
xi_lin with zeta1-zeta4@0;
zeta1 with zeta2-zeta4@0;
zeta2 with zeta3-zeta4@0;
zeta3 with zeta4@0;
MODEL CONSTRAINT:
! lambdaid is double lambda i.
lambda2d = lambda2*2;
lambda3d = lambda3*2;
! lambdait is triple lambda i.
lambda2t = lambda2*3;
lambda3t = lambda3*3;
OUTPUT: sampstat stdyx;
```

Model 2c: GSGM, α, λ, γ-MI

filename: 2c_GSGM_LinearGrowth_AlphaLamGamInv.inp

```
TITLE: Generalized Second-Order Multiple-Indicator Growth Model.
       Measurement invariance assumed on alpha and lambda.
       Measurement invariance assumed on gamma.
       Linear Growth.
       First-order model representation.
```

```
          (after Schmid-Leiman Transformation).
DATA: file = Anxiety_Data_3_Indicators_4_Waves.dat;
VARIABLE:
names = grpID
        y11 y21 y31
        y12 y22 y32
        y13 y23 y33
        y14 y24 y34;
usevariables = y11 y21 y31
               y12 y22 y32
               y13 y23 y33
               y14 y24 y34;
missing = all(-99);
MODEL:
! Common xi. This is the trait intercept.
! Set the first trait loading (lambda) to 1.
! Estimate the other two.
! This assumes Measurement Invariance on lambda.
! Do this if the lambdas have to be the same
! for a given test across testing occasions.
xi_int by y11@1 y21 y31 (lambda1-lambda3)
          y12@1 y22 y32 (lambda1-lambda3)
          y13@1 y23 y33 (lambda1-lambda3)
          y14@1 y24 y34 (lambda1-lambda3);
! Estimate the mean of the trait factor.
! (zero by default in Mplus).
[xi_int*];
! Variance of the trait factor (xi_int).
xi_int (xi_intv);
! Growth Model. This is the trait slope.
! Wave 1 is missing, because lambda would be multiplied by 0.
! Multiply by 1*lambda for the second wave, by 2*lambda for
! the third, and 3*lambda for the fourth.
xi_lin by y12@1 (lambda1)
          y22   (lambda2)
          y32   (lambda3)
          y13@2 (lambda1d)
          y23   (lambda2d)
          y33   (lambda3d)
          y14@3 (lambda1t)
          y24   (lambda2t)
          y34   (lambda3t);
! Estimate the mean of the growth factor.
! (zero by default in Mplus).
[xi_lin*];
```

```
! Variance of the growth factor (xi_lin).
xi_lin (xi_linv);
! These loadings are NOT the same as the trait
! loadings given above. This is the difference
! between the GSGM and the SGM.
! Assume measurement invariance of the state residual
! factor loadings (gamma_it=gamma_is).
zeta1 by y11@1 (gamma1)
         y21   (gamma2)
         y31   (gamma3);
zeta2 by y12@1 (gamma1)
         y22   (gamma2)
         y32   (gamma3);
zeta3 by y13@1 (gamma1)
         y23   (gamma2)
         y33   (gamma3);
zeta4 by y14@1 (gamma1)
         y24   (gamma2)
         y34   (gamma3);
! Set intercepts of state residual factors to zero.
! (would otherwise be estimated as the default in Mplus)
[zeta1-zeta4@0];

! Delta error variances on each zeta.
zeta1 (zeta1var);
zeta2 (zeta2var);
zeta3 (zeta3var);
zeta4 (zeta4var);
! Constant intercepts (alpha_it).
! Set the intercept on the first indicator
! (alpha1) to 0, and estimate the other
! two (alpha2, alpha3).
! This assumes alpha-MI.
[y11@0]; [y21*] (alpha2); [y31*] (alpha3);
[y12@0]; [y22*] (alpha2); [y32*] (alpha3);
[y13@0]; [y23*] (alpha2); [y33*] (alpha3);
[y14@0]; [y24*] (alpha2); [y34*] (alpha3);
! Estimate the Error Variances.
y11 (ev11); y21 (ev21); y31 (ev31);
y12 (ev12); y22 (ev22); y32 (ev32);
y13 (ev13); y23 (ev23); y33 (ev33);
y14 (ev14); y24 (ev24); y34 (ev34);
! Non-admissible correlations.
xi_int with zeta1-zeta4@0;
xi_lin with zeta1-zeta4@0;
```

```
zeta1 with zeta2-zeta4@0;
zeta2 with zeta3-zeta4@0;
zeta3 with zeta4@0;
MODEL CONSTRAINT:
! lambdaid is double lambda i.
lambda2d = lambda2*2;
lambda3d = lambda3*2;
! lambdait is triple lambda i.
lambda2t = lambda2*3;
lambda3t = lambda3*3;
OUTPUT: sampstat stdyx;
```

Model 3a: ISGM

filename: 3a__ISGM__LinearGrowth.inp

```
TITLE: Indicator-Specific Growth Model.
       Linear Growth.
       First-order model representation.
DATA: file = Anxiety_Data_3_Indicators_4_Waves.dat;
VARIABLE:
names = grpID
        y11 y21 y31
        y12 y22 y32
        y13 y23 y33
        y14 y24 y34;
usevariables = y11 y21 y31
               y12 y22 y32
               y13 y23 y33
               y14 y24 y34;
missing = all(-99);
MODEL:
! One xi (trait intercept factor) for each indicator.
! All loadings are set to 1, because of the
! indicator-specificity. See derivation.
xi_int_1 by y11@1 y12@1 y13@1 y14@1;
xi_int_2 by y21@1 y22@1 y23@1 y24@1;
xi_int_3 by y31@1 y32@1 y33@1 y34@1;
! Estimate the trait factor means
! (zero by default in Mplus).
[xi_int_1* xi_int_2* xi_int_3*];
! Variance on xi_int_i.
xi_int_1 (xi_int1v);
xi_int_2 (xi_int2v);
xi_int_3 (xi_int3v);
! Growth Model. These are the trait slopes.
```

```
! Linear Growth. No loading on the first
! indicator as this would be 0.
xi_lin_1 by y12@1 y13@2 y14@3;
xi_lin_2 by y22@1 y23@2 y24@3;
xi_lin_3 by y32@1 y33@2 y34@3;
! Estimate the growth factor means
! (zero by default in Mplus).
[xi_lin_1* xi_lin_2* xi_lin_3*];
! Estimate the growth factor variances.
xi_lin_1 (xi_lin1v);
xi_lin_2 (xi_lin2v);
xi_lin_3 (xi_lin3v);
! Occasion-specific (common) state residual
! factors (zeta_t).
! Do not assume measurement invariance of the state residual
! factor loadings (gamma_it><=gamma_is).
zeta1 by y11@1
         y21 (gamma21)
         y31 (gamma31);
zeta2 by y12@1
         y22 (gamma22)
         y32 (gamma32);
zeta3 by y13@1
         y23 (gamma23)
         y33 (gamma33);
zeta4 by y14@1
         y24 (gamma24)
         y34 (gamma34);
! Set intercepts of state residual factors to zero.
! (would otherwise be estimated as the default in Mplus)
[zeta1-zeta4@0];
! Variances of zeta.
zeta1 (zeta1var);
zeta2 (zeta2var);
zeta3 (zeta3var);
zeta4 (zeta4var);
! Intercepts factors (alpha loadings).
! Set the intercept loadings (alpha_it) on
! all indicators to 0 as required by the model.
! Intercepts.
! Set to 0, as required by the model.
[y11@0 y21@0 y31@0];
[y12@0 y22@0 y32@0];
[y13@0 y23@0 y33@0];
[y14@0 y24@0 y34@0];
```

```
! Estimate the Error Variances.
y11 (ev11); y21 (ev21); y31 (ev31);
y12 (ev12); y22 (ev22); y32 (ev32);
y13 (ev13); y23 (ev23); y33 (ev33);
y14 (ev14); y24 (ev24); y34 (ev34);
! Non-admissible correlations.
xi_int_1-xi_int_3 with zeta1-zeta4@0;
xi_lin_1-xi_lin_3 with zeta1-zeta4@0;
zeta1 with zeta2-zeta4@0;
zeta2 with zeta3-zeta4@0;
zeta3 with zeta4@0;
OUTPUT: sampstat stdyx;
```

Model 3b: ISGM, γ-MI

filename: 3b__ISGM__LinearGrowth__GamInv.inp

```
TITLE: Indicator-Specific Growth Model.
       Measurement invariance assumed on gamma.
          (gamma_it = gamma_is for all i,t,s)
       Linear Growth.
       First-order model representation.
DATA: file = Anxiety_Data_3_Indicators_4_Waves.dat;
VARIABLE:
names = grpID
        y11 y21 y31
        y12 y22 y32
        y13 y23 y33
        y14 y24 y34;
usevariables = y11 y21 y31
               y12 y22 y32
               y13 y23 y33
               y14 y24 y34;
missing = all(-99);
MODEL:
! One xi (trait intercept factor) for each indicator.
! All loadings are set to 1, because of the
! indicator-specificity. See derivation.
xi_int_1 by y11@1 y12@1 y13@1 y14@1;
xi_int_2 by y21@1 y22@1 y23@1 y24@1;
xi_int_3 by y31@1 y32@1 y33@1 y34@1;
! Estimate the trait factor means
! (zero by default in Mplus).
[xi_int_1* xi_int_2* xi_int_3*];
! Variance on xi_int_i.
xi_int_1 (xi_int1v);
```

```
xi_int_2 (xi_int2v);
xi_int_3 (xi_int3v);
! Growth Model. These are the trait slopes.
! Linear Growth. No loading on the first
! indicator as this would be 0.
xi_lin_1 by y12@1 y13@2 y14@3;
xi_lin_2 by y22@1 y23@2 y24@3;
xi_lin_3 by y32@1 y33@2 y34@3;
! Estimate the growth factor means
! (zero by default in Mplus).
[xi_lin_1* xi_lin_2* xi_lin_3*];
! Estimate the growth factor variances.
xi_lin_1 (xi_lin1v);
xi_lin_2 (xi_lin2v);
xi_lin_3 (xi_lin3v);
! Occasion-specific (common) state residual
! factors (zeta_t).
! Assume Measurement invariance of the state residual
! factor loadings (gamma_it=gamma_is).
zeta1 by y11@1
         y21 (gamma2)
         y31 (gamma3);
zeta2 by y12@1
         y22 (gamma2)
         y32 (gamma3);
zeta3 by y13@1
         y23 (gamma2)
         y33 (gamma3);
zeta4 by y14@1
         y24 (gamma2)
         y34 (gamma3);
! Set intercepts of state residual factors to zero.
! (would otherwise be estimated as the default in Mplus)
[zeta1-zeta4@0];
! Variances of zeta.
zeta1 (zeta1var);
zeta2 (zeta2var);
zeta3 (zeta3var);
zeta4 (zeta4var);
! Intercepts factors (alpha loadings).
! Set the intercept loadings (alpha_it) on
! all indicators to 0 as required by the model.
! Intercepts.
! Set to 0, as required by the model.
[y11@0 y21@0 y31@0];
```

```
[y12@0 y22@0 y32@0];
[y13@0 y23@0 y33@0];
[y14@0 y24@0 y34@0];
! Estimate the Error Variances.
y11 (ev11); y21 (ev21); y31 (ev31);
y12 (ev12); y22 (ev22); y32 (ev32);
y13 (ev13); y23 (ev23); y33 (ev33);
y14 (ev14); y24 (ev24); y34 (ev34);
! Non-admissible correlations.
xi_int_1-xi_int_3 with zeta1-zeta4@0;
xi_lin_1-xi_lin_3 with zeta1-zeta4@0;
zeta1 with zeta2-zeta4@0;
zeta2 with zeta3-zeta4@0;
zeta3 with zeta4@0;
OUTPUT: sampstat stdyx;
```

Appendix H

Monte-Carlo Simulation for the Indicator-specific Growth Model

A Monte-Carlo simulation study was conducted to study the behavior of the indicator-specific growth model (ISGM) under a variety of conditions. Population parameters were generated for five different conditions that were fully crossed. These included the number of time-points (time; 3 levels), sample size (N; 5 levels), consistency (con; 3 levels), intercept and slope correlations (corr; 3 levels), and effect-size for mean change based on Cohen's d (cohen; 4 levels). The size of this simulation was thus $(3 \times 5 \times 3 \times 3 \times 4) = 540$ cells.

In our choice of parameter values and other conditions, we included conditions that could be seen as somewhat extreme. This was done in order to "test the limits" of the ISGM, that is, to examine the conditions under which the model might fail. Specifically, we included low sample size conditions (N=100; 150; 200), conditions of low consistency (.2;.5), and high (.9) as well as low (.1) intercept/slope factor correlations. These parameter values, especially the low consistency and high correlation conditions were expected to bring the model to its limits, given that low consistency means weakly-defined trait factors and high correlations could make the model prone to improper solutions (correlation estimates > 1.0).

Algorithm 1 Generation of ISGM Population Parameters for the Simulation

for $t = 1$ to {time}
  $\text{var}(\zeta_t) = 1.0$
end
for i=1 to 3
  $\text{var}(Y_{i1}) = 1$
  $\text{Con}(\tau_{i1}) = \{\text{Con}\} + \sim \mathcal{N}(0, 0.025)$
  $\text{Rel}(Y_{i1}) = 0.73 + \sim N(0, 0.025)$
  $\text{var}(\varepsilon_{i1}) = 1 - \text{Rel}(Y_{i1})$
  $\text{var}(\tau_{i1}) = \text{var}(Y_{i1}) - \text{var}(\varepsilon_{i1})$
  $\text{var}(\xi_{\text{int}_i}) = \text{Con}(\tau_{i1}) \cdot \text{var}(\tau_{i1})$
  $\text{var}(\xi_{\text{lin}_i}) = 0.05 \cdot \text{var}(\xi_{\text{int}_i})$
  $\gamma_i = \sqrt{\text{var}(\tau_{i1}) - \text{var}(\xi_{\text{int}_i})}$
  for $t = 2$ to {time}
    $\text{var}(\tau_{it}) = \text{var}(\xi_{\text{int}_i}) + (t-1)^2 \text{var}(\xi_{\text{lin}_i}) + \gamma_i^2 \text{var}(\zeta_t)$
    $\text{var}(\varepsilon_{it}) = \text{var}(\tau_{it}) \cdot (1 - \text{Rel}(Y_{i1})) + \sim \mathcal{N}(0, 0.03)$
    $\text{var}(Y_{it}) = \text{var}(\tau_{it}) + \text{var}(\varepsilon_{it})$
  end
end
for $i = 1$ to 3
  for $j = 1$ to 3
    $\text{cov}\left(\xi_{\text{int}_i}, \xi_{\text{int}_j}\right) = \{\text{corr}\} \cdot \sqrt{\text{var}(\xi_{\text{int}_i})}\sqrt{\text{var}\left(\xi_{\text{int}_j}\right)}$
    $\text{cov}\left(\xi_{\text{lin}_i}, \xi_{\text{lin}_j}\right) = \{\text{corr}\} \cdot \sqrt{\text{var}(\xi_{\text{lin}_i})}\sqrt{\text{var}\left(\xi_{\text{lin}_j}\right)}$
  end
end

Table H1
Simulation Parameters Used to Identify Population Values for the ISGM.

Parameter (abbreviation)	Levels	Description
Number of time points (time)	$= \{3, 4, 5\}$	
Sample size (N)	$= \{100, 150, 200, 300, 500\}$	
Average Consistency (con)	$\approx \{0.8, 0.5, 0.2\}$	$\frac{\sum_{i=1}^{m}\sum_{t=1}^{n}[\text{Con}(\tau_{it})]}{m \cdot n}$, $\text{Con}(\tau_{it}) = \frac{\text{var}(\xi_{\text{int}_i}) + (t-1)^2\,\text{var}(\xi_{\text{lin}_i})}{\text{var}(\tau_{it})}$
Intercept and slope correlations (corr)	$= \{0.9, 0.5, 0.1\}$	$\text{corr}\left(\xi_{\text{int}_i}, \xi_{\text{int}_j}\right), \text{corr}\left(\xi_{\text{lin}_i}, \xi_{\text{lin}_j}\right)$
Effect-size based on Cohen's d (cohen)	$= \{0.0, 0.2, 0.5, 0.8\}$	$\frac{E(\xi_{\text{lin}_i})}{\sqrt{\text{var}(\xi_{\text{lin}_i})}}$
Average Reliability (rel)	$\approx \{0.8\}$	$\frac{\sum_{i=1}^{m}\sum_{t=1}^{n}[\text{Rel}(Y_{it})]}{m \cdot n}$, $\text{Rel}(Y_{it}) = \frac{\text{var}(\xi_{\text{int}_i}) + (t-1)^2\,\text{var}(\xi_{\text{lin}_i}) + \gamma_{it}^2\,\text{var}(\zeta_t)}{\text{var}(Y_{it})}$
Trait-growth correlations	$\{0.0\}$	$\text{corr}\left(\xi_{\text{int}_i}, \xi_{\text{lin}_j}\right)$ for all i, j
Trait means and indicator intercepts	$\{0.0\}$	$E(\xi_{\text{int}_i}), \alpha_i$ for all i
Trait/growth variance ratio	$\{0.05\}$	$\frac{\text{var}(\xi_{\text{lin}_i})}{\text{var}(\xi_{\text{int}_i})}$ for all i

Note. Parameters not listed were identified through constraints with other parameters as given in Algorithm 1. $i = 1, \ldots, j, \ldots, m$, indicates the manifest variable or indicator. $t = 1, \ldots, s, \ldots n$ indicates the time point.

Method

There were two main steps involved in developing the simulation inputs, a) generation of the population parameters, and b) specification of the models. The general idea in generating population parameters was to use the approximate parameter levels shown in Table H1 for consistency, correlations, reliability, and cohen's d, but allow for some variation across conditions to obtain a more realistic scenario. Throughout this process, it was necessary to make some arbitrary assumptions concerning the value to which particular parameters were set. For example, observed variances for the first time point (Y_{i1}) were set to 1.0, as were state residual variances $\text{var}(\zeta_t)$. The relations among model parameters, indicator reliability, consistency, total observed variance, and some randomness were all used to determine the true values for population parameters. This led to the development of Algorithm 1, which was implemented in Matlab, and led to population values within the desired range.

The Monte Carlo facility of Mplus was then used for the analysis both to generate samples from the given population values, and to fit the corresponding correctly specified latent growth models to the simulated data over 1,000 replications per cell. The total number of models fit for this simulation was thus 540,000. For these models, invariance of state residual factor loadings was assumed, and the first factor loading (γ_1) was fixed to the known population value for each time point as required for model identification. Correlations/covariances between trait and slope factors were not estimated, but were fixed to 0. This specification was chosen mostly to simplify the variance decomposition, leading to a simpler algorithm for the generation of the remaining population parameters. Expected values for latent trait and slope factors $[E(\xi_{\text{int}_i}), E(\xi_{\text{lin}_i})]$, trait and intercept factor variances $[\text{var}(\xi_{\text{int}_i}), \text{var}(\xi_{\text{lin}_i})]$, state residual variances $[\text{var}(\zeta_t)]$ state residual factor loadings for non-reference indicators (γ_2, γ_3), error variances $[\text{var}(\varepsilon_{it})]$, intercept correlations $\left[\text{corr}\left(\xi_{\text{int}_i}, \xi_{\text{int}_j}\right)\right]$, and slope correlations $\left[\text{corr}\left(\xi_{\text{lin}_i}, \xi_{\text{lin}_j}\right)\right]$ were all estimated.

Criteria for Evaluating the Performance of the Models

Six criteria were used to evaluate the performance of the ISGM: 1) non-convergence, 2) improper solutions, 3) χ^2 distribution approximation, 4) parameter estimation bias, 5) standard error bias, and 6) coverage. These criteria are discussed below.

Non-convergence. For non-convergence, we recorded the number of replications for which the estimation process did not converge after 1000 iterations. The percent convergence was then computed, which is given by

$$\%\text{convergence} = \frac{\text{number of replications with no convergence after 1000 iterations}}{\text{number of replications requested}} \times 100.$$

Improper Solutions. Two types of improper solutions were evaluated. The first type refers to the number of replications with a non-positive definite (npd) residual covariance matrix, Θ (theta). These will be referred to as Θ warnings. The second type refers to the number of replications with a npd latent variable covariance matrix, Ψ (psi). These will be referred to as Ψ warnings.

χ^2 Distribution Approximation. The adequacy of the χ^2 distribution approximation was assessed by comparing the observed χ^2 distribution across replications with the theoretical χ^2 distribution. This helps to evaluate whether the empirical χ^2 value is appropriate to assess model fit. If the theoretical χ^2 distribution is not sufficiently approximated in a certain cell of the simulation design, this would indicate that the theoretical χ^2 distribution cannot be used in this case to obtain a valid p-value for the empirical χ^2 statistic.

In order to get a single representative statistic for each model, the fit was assessed by creating a mean of the observed differences,

$$\chi^2\text{stat} = \frac{\sum\left|\chi^2_{\text{expected}} - \chi^2_{\text{observed}}\right|}{N},$$

where χ^2_{expected} is the proportion expected, and χ^2_{observed} is the corresponding proportion observed. The sum in the numerator is made over thirteen expected proportions given in Mplus: .99, .98, .95, .90, .80, .70, .50, .30, .20, .10, .05, .02, and .01; N in the denominator is equal to 13.

Parameter Estimation Bias. The parameter estimation bias is a measure of the accuracy with which the true population parameters are reproduced and is given by

$$\text{peb} = \frac{M_p - e_p}{e_p},$$

where M_p is the mean of the parameter estimates over all replications and e_p is the parameter value in the population. Values of less that .10 for peb are generally considered acceptable.

Standard Error Bias . The standard error bias is a measure of the reliability of the reported standard error values, and is useful in determining the appropriateness of tests of significance of the model parameters. High bias values for the standard error indicate that significance tests may be unreliable (increased type I or type II errors). The standard error bias is given by

$$\text{seb} = \frac{M_{\text{SE}} - \text{SD}_p}{\text{SD}_p},$$

where M_{SE} is the mean of the standard errors over all replications, and SD_p is the standard deviation of the estimated model parameters over all replications.

Coverage. Coverage refers to the proportion of replications for which the 95% confidence interval actually contains the true population value. The nominal value for coverage is .95.

Results

Non-convergence. The total number of non-converged replications was 173,305, which represented 32% of the 540,000 total replications requested. Although this number seems large, the distribution of non-converged solutions was heavily skewed, with 37% of the cells having fewer than 10% non-converged solutions. The simulation results were further examined using a regression analysis to identify the factors that were most strongly related to non-convergence. The regression models and R^2 values are shown in Table H2. The most important factors were the number of time points, the level of trait consistency, and sample size. This can be seen in Figure H1, which shows observed proportions of non-converged solutions grouped by condition. While the proportion of non-converged solutions was unacceptably high (.91) for the worst condition tested (sample size = 100, consistency = .2, and only 3 time-points), non-convergence dropped below .001 (i.e., 0.1%) when consistency was above .5 and five time-points were present.

Improper Solutions. The total number of improper solutions due to npd Θ matrices was 910, which is less than 0.2%. The overall number of solutions with npd Ψ matrices was much higher (132,954 or 25%), and sometimes exceeded 50% of the total number of replications requested for a cell. Of the 132,954 replications with a Ψ warning message, 117,335 (88%) contained at least one improper parameter estimate, and 15,619 (12%) had none. The absence of improper estimates when Ψ warnings are present indicates that the Ψ warning is due to a linear dependency among parameters. There were a total of 191,864 improper parameter estimates, since there were multiple improper parameter estimates for some replications. Of the 191,864 improper parameter estimates, 177,805 (93%) were out of range (i.e., >1.0) slope correlations, 11,761 (6%) were improper (i.e., >1.0) intercept correlations, and 2,298 (1%) were accounted for by other latent variable parameters.

Table H2

Regression Results for Non-convergence, Improper Solutions, and χ^2 statistic.

Criteria	Model	R^2
(Non-)convergence		
	converged ~ time	.51
	converged ~ N	.08
	converged ~ con	.34
	converged ~ cohen	.00
	converged ~ corr	.00
	converged ~ time + con	.85
	converged ~ time + con + N	.93
Improper solutions: Θ		
	Θ ~ time	.08
	Θ ~ N	.20
	Θ ~ con	.06
	Θ ~ cohen	.00
	Θ ~ corr	.02
	Θ ~ N + corr	.22
	Θ ~ N + corr + time	.30
	Θ ~ N*corr*time - N:corr:time	.51
	Θ ~ N*corr*time	.52
Improper solutions: Ψ		
	Ψ ~ time	.02
	Ψ ~ N	.00
	Ψ ~ con	.00
	Ψ ~ cohen	.00
	Ψ ~ corr	.62
	Ψ ~ time + corr	.65
	Ψ ~ time*corr	.71
χ^2 statistic		
	χ^2 stat ~ time	.46
	χ^2 stat ~ N	.35
	χ^2 stat ~ con	.01
	χ^2 stat ~ cohen	.00
	χ^2 stat ~ corr	.00
	χ^2 stat ~ time + N	.81
	χ^2 stat ~ time*N	.97

Note. Regression models are presented in Wilkinson-Rogers (1973) notation (for example, a*b indicates that terms a, b, and the product a*b were used in the regression; a + b indicates that only a and b were used; a*b - a:b is equivalent to a + b, as - a:b denotes that product a*b was excluded). converged = the number of replications that converged, Ψ = number of replications with an improper residual covariance matrix, Θ = number of replications with an improper latent variable covariance matrix, χ^2 stat = a composite measure of the adequacy of the χ^2 distribution approximation, con = consistency, N = sample size, time = the number of time points, cohen = effect size for mean change based on Cohen's d.

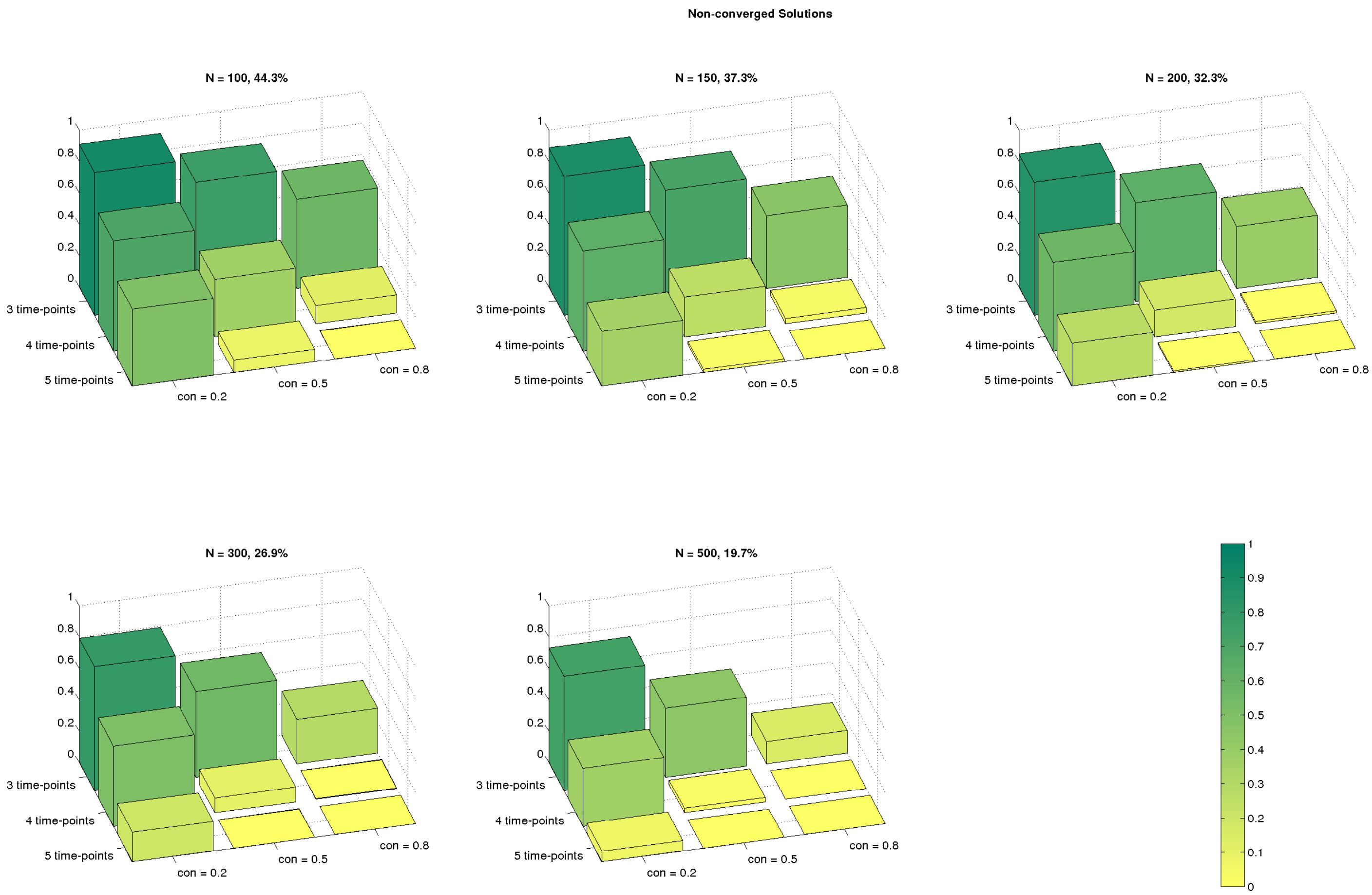


Figure H1. Proportion of non-converged replications, grouped by condition; N = sample size, con = consistency. The proportion of non-converged replications tended to decrease as sample size, consistency, and the number of time points increase.

The results of a regression analysis on each of these warning messages is shown in Table H2. These analyses revealed that while uncommon overall, Θ warnings occurred most often when the latent correlations, number of time points, and sample size were lowest. The number of Ψ warnings was highest when the latent correlations were high (0.9), which is expected, since high correlations make the occurrence of correlation estimates greater than 1.0 due to sampling error more likely, especially in smaller samples. This behavior can be seen graphically in Figure H2, which shows the proportion of improper solutions grouped by the number of time points and level of correlation.

χ^2 **Distribution Approximation.** The results for the χ^2 distribution approximation indicated that the theoretical χ^2 distribution was generally well-approximated, with an average difference between expected and observed proportions below 0.1 in all conditions. Table H2 shows the R^2 fit statistics for the regression analysis, which indicated that the

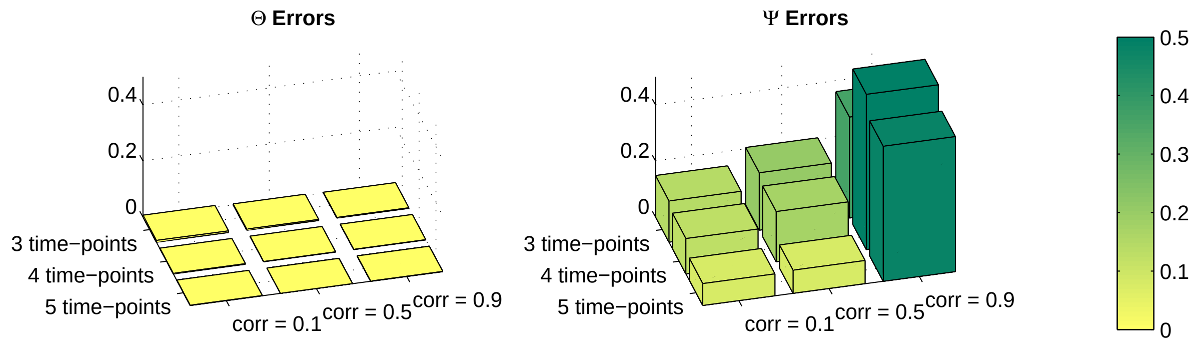


Figure H2. Solutions with an improper residual covariance matrix, Θ (theta) or an improper latent variable covariance matrix, Ψ (psi).

number of time points and the sample size were the best predictors for a deviation between the theoretical and empirical χ^2 distributions. The largest magnitude of differences between expected and observed proportions occurred when the sample size was low, and the number of time points was high. This dependency can be seen in Figure H3.

Parameter Estimation Bias. The performance criteria discussed previously (non-convergence, improper solutions, and χ^2 statistic) all resulted in a single metric for each model. In contrast, the final three criteria (peb, seb, and coverage) provide a performance value for each parameter estimated in the model. The average peb for most parameters was less than 0.1, which can be seen in Figure H4A. This indicates that estimates of these parameters tended to be well approximated. The peb for the expected value of the trait

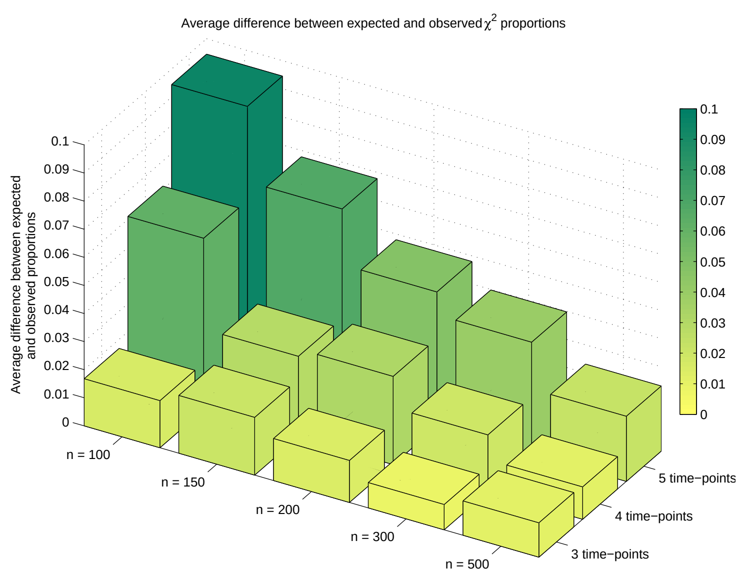
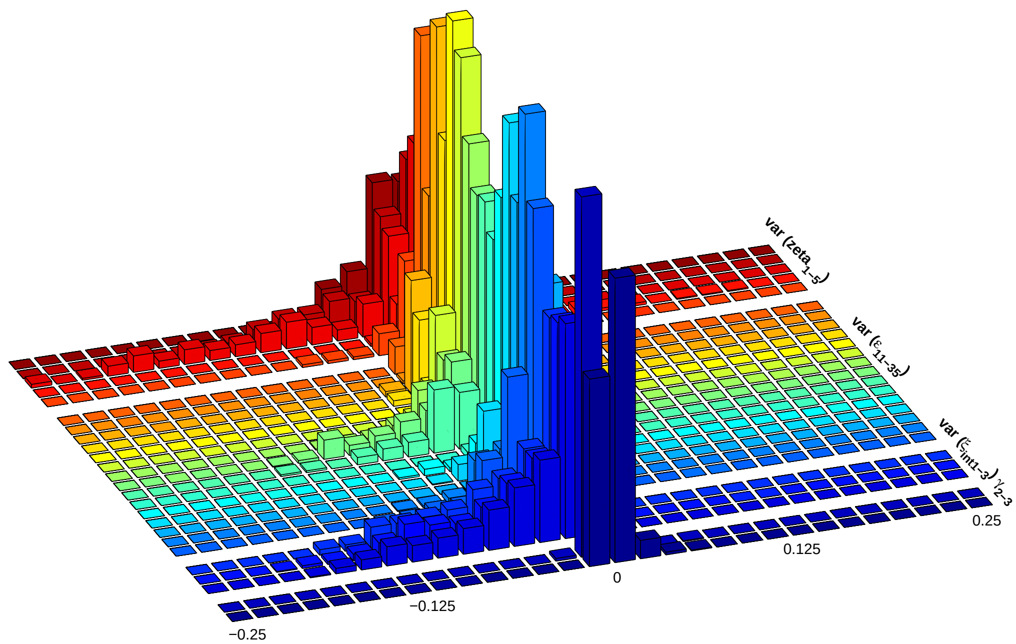


Figure H3. χ^2 Distribution Approximation. The average absolute distance between the expected and observed proportions of the χ^2 distribution.

A



B

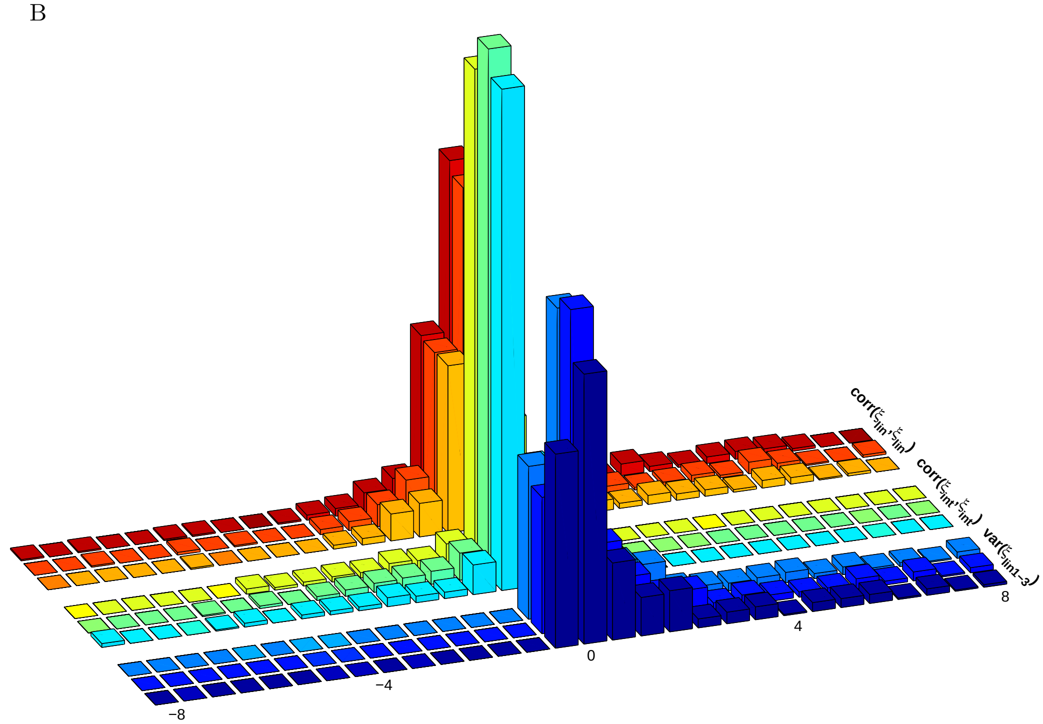


Figure H4. Parameter estimation bias proportion histograms. A: The parameters (front to back) are: γ_2, γ_3, var (ξ_{int_1}), var (ξ_{int_2}), var (ξ_{int_3}), var (ε_{11}), var (ε_{21}), var (ε_{31}), var (ε_{12}), var (ε_{22}), var (ε_{32}), var (ε_{13}), var (ε_{23}), var (ε_{33}), var (ε_{14}), var (ε_{24}), var (ε_{34}), var (ε_{15}), var (ε_{25}), var (ε_{35}), var (ζ_1), var (ζ_2), var (ζ_3), var (ζ_4), and var (ζ_5). B: The parameters (front to back) are: var (ξ_{lin_1}), var (ξ_{lin_2}), var (ξ_{lin_3}), corr $(\xi_{\text{int}_1}, \xi_{\text{int}_2})$, corr $(\xi_{\text{int}_1}, \xi_{\text{int}_3})$, corr $(\xi_{\text{lin}_1}, \xi_{\text{lin}_2})$, corr $(\xi_{\text{lin}_1}, \xi_{\text{lin}_3})$, corr $(\xi_{\text{lin}_2}, \xi_{\text{lin}_3})$.

intercept and growth factors could not be estimated since the population values were set to 0. However, these parameters were also approximated well; the mean difference between the population and estimated values was 0.00, and the standard deviation was 0.003. The peb for the intercept correlations [corr $\left(\xi_{\text{int}_i}, \xi_{\text{int}_j}\right)$], slope correlations [corr $\left(\xi_{\text{lin}_i}, \xi_{\text{lin}_j}\right)$], and growth factor variances [var (ξ_{lin_i})] was higher, as seen in Figure H4B. The average peb of the intercept correlations was -0.26, with a standard deviation of 0.80. The average peb for the slope correlations was 0.33 with a standard deviation of 1.51. The average peb for the growth factor variances was positively skewed, with a mean of 0.87 and a standard deviation of 1.55.

A separate regression analysis was performed to examine the most important factors causing peb for each parameter. The R^2 values resulting from this regression are shown in Figure H5. Due to the high peb values observed for the growth factor variances, the intercept correlations, and the slope correlations, these were investigated further to determine the conditions under which high peb levels occurred. For this, an overall peb metric for each parameter set was made by averaging the absolute bias value for each parameter with other model parameters from that set. The R^2 fit for the regression with the combined peb metrics is shown in Table H3. The peb for the growth factor variances was highest the lower the level of consistency, sample size, and number of time points. This is demonstrated

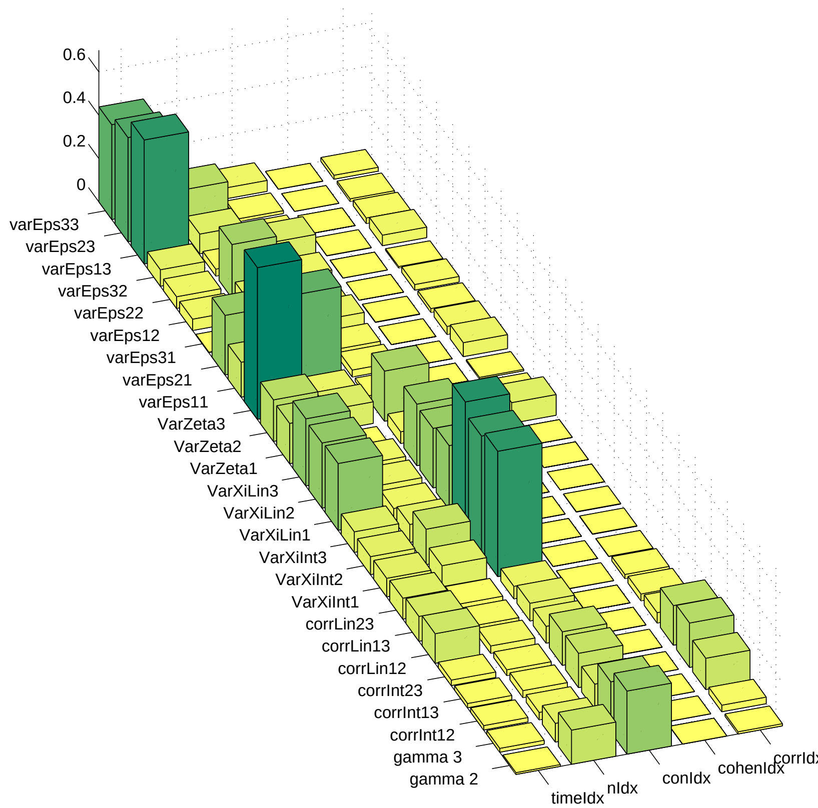


Figure H5. R^2 values for regression of parameter estimation bias by time, N, cohen, and corr. Parameters not estimated in all conditions [e.g., var (ε_{14-35}), var (ζ_{4-5})] were omitted.

Table H3

Regression Results for peb, seb, and Coverage

Criteria	Model	R^2		
		$\text{var}(\xi_{\text{lin}_i})$	$\text{corr}(\xi_{\text{int}_i}, \xi_{\text{int}_j})$	$\text{corr}(\xi_{\text{lin}_i}, \xi_{\text{lin}_j})$
Parameter estimate bias				
	peb ~ time	.31	.02	.08
	peb ~ N	.06	.03	.01
	peb ~ con	.29	.15	.12
	peb ~ cohen	.00	.00	.00
	peb ~ corr	.00	.18	.32
	peb ~ time*con	.82	.20	.23
	peb ~ time*con + N	.89	.24	.24
	peb ~ time*con*N	.99	.30	.27
	peb ~ time*con + corr	.83	.38	.56
	peb ~ time*con*corr	.83	.69	.86
Standard Error Bias				
	seb ~ time	0.48	.01	.00
	seb ~ N	0.04	.01	.02
	seb ~ con	0.41	.03	.01
	seb ~ cohen	0.00	.01	.01
	seb ~ corr	0.00	.00	.00
	seb ~ time*con	0.93	.06	.02
	seb ~ time*con + N	0.96	.07	.04
	seb ~ time*con*N	0.99	.15	.12
	seb ~ time*con + corr	0.93	.06	.02
	seb ~ time*con*corr	0.93	.10	.05
Coverage				
	cover ~ time		.00	.14
	cover ~ N		.02	.03
	cover ~ con		.07	.05
	cover ~ cohen		.00	.00
	cover ~ corr		.77	.42
	cover ~ corr*con		.91	.54
	cover ~ corr*time		.78	.66
	cover ~ corr*con*time		.92	.88
	cover ~ corr*con*N		.98	.60

Note. Regression models are presented in Wilkinson-Rogers (1973) notation. peb = parameter estimation bias, seb = standard error bias, cover = the proportion of replications for which the 95% confidence interval contains the true population value, con = consistency, N = sample size, time = the number of time points, cohen = effect size for mean change based on Cohen's d.

graphically in Figure H6. The peb for the intercept and slope correlations behaved similarly. Parameter estimate bias was highest the lower the actual level of correlation, consistency, and number of time points. This is depicted in Figure H7, which also demonstrates that intercept correlation estimates were less biased than the slope correlations.

Standard Error Bias . Standard error bias for the estimated parameters is shown in Figures H8 and H9. The seb for the parameters in Figure H8 was low, suggesting that tests of significance for these model parameters were reliable. As with peb, three sets of parameters were the exception to this rule: the intercept correlations, slope correlations, and growth factor variances. Of these, the standard error of the growth factor variances was the least biased, with a mean of 0.21 and a standard deviation of 0.21. The mean seb for intercept correlations was 0.25, and the standard deviation was 2.56, while the mean seb for slope correlations was 24.8, with a standard deviation of 263.

The seb regression results for most parameters are shown in Figure H9. From this plot, we see that for most parameters, seb is predicted by the number of time points and the sample size. For the trait and growth variances, consistency is also a strong predictor. The conditions which produced high seb values for the correlations and growth factor variances was again investigated further using a combined seb metric for each parameter set. The models and R^2 values are shown in Table H3.

Differences between conditions with respect to seb for the growth factor variance was explained by consistency, sample size, and number of time points in the expected way, as shown in Figure H10. As indicated by the regression results, no clear set of conditions was found that linearly predicted the seb for the intercept and slope correlations, which was large for most cells. However, the condition with the highest number of time points ($t = 5$), highest consistency (Con $= 0.8$), and highest sample size ($N = 500$) produced a mean seb of 0.01 with a standard deviation of 0.002 for both the intercept and slope correlations, which can be seen as good.

Coverage. The average coverage for each parameter is shown in Figure H11. As depicted in the graph, coverage values for most parameters were well approximated. The intercept and slope correlations were the only parameters for which the coverage values were low enough to be of concern. Coverage for intercept correlations were the lowest, followed by slope correlations. Regression was again performed on the coverage for each separate parameter, as shown in Figure H12. Coverage was most heavily dependent on the level of correlation for both intercept and slope-slope correlations, which can also be seen from the regression values shown in Table H3. Figure H13 shows the average coverage by condition for intercept correlations, and Figure H14 shows the average coverage for slope correlations. For both of these sets of parameters, coverage was highest when the population correlation was 0.5. Coverage was low when the correlation was either very high or very low. The second most important factor for both sets of correlations was consistency. Coverage for intercept correlations was higher when consistency was high. For slope correlations the same general trend is true for the 5 time point condition and also for the condition with 4 time points. Coverage for intercept correlations did not depend on the number of time points, while coverage for the slope correlations did. Sample size was more important in the coverage for intercept correlations than for slope correlations.

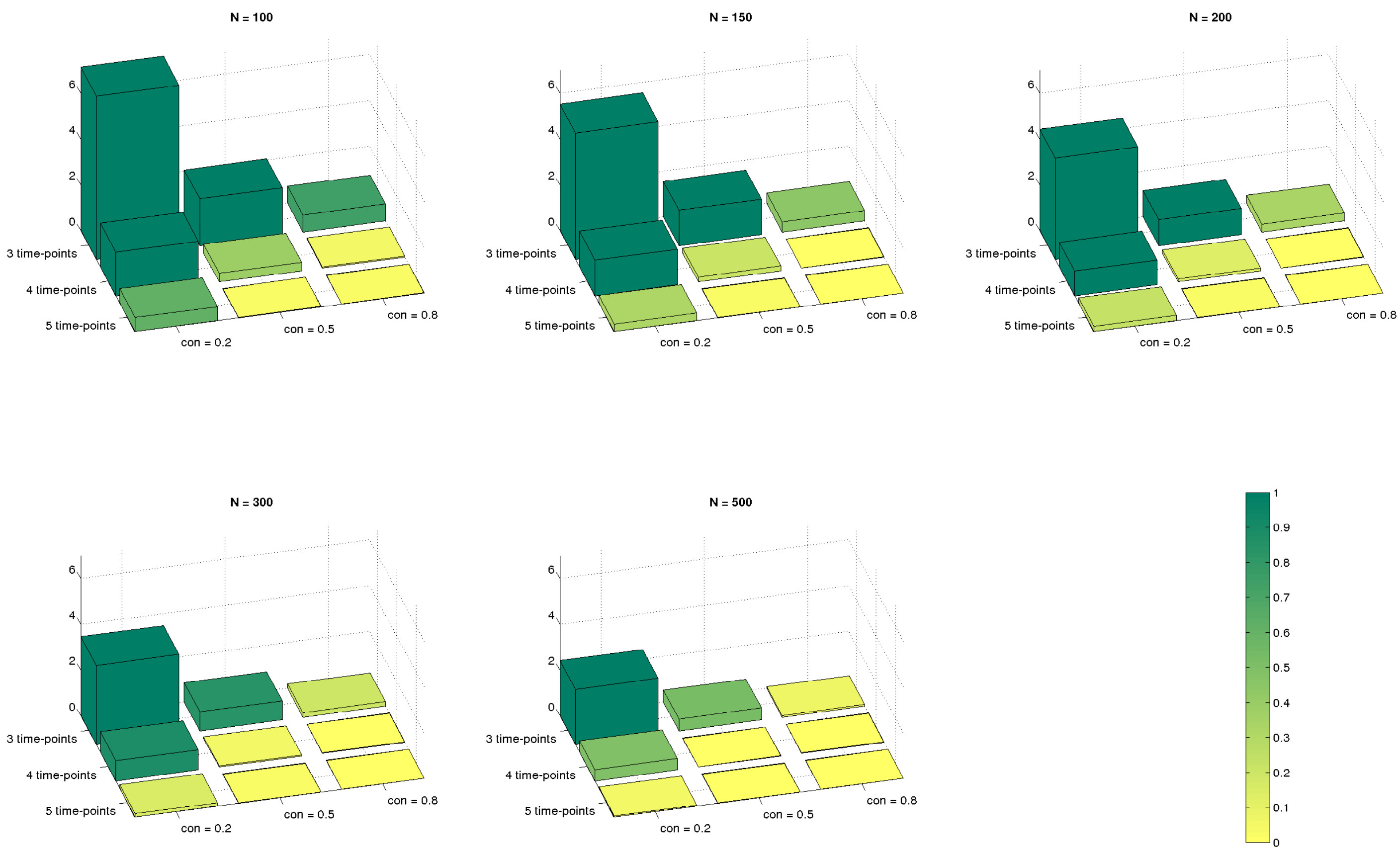


Figure H6. Parameter estimation bias for growth factor variances, var $(\xi_{\text{lin}_{1-3}})$.

Summary and Discussion

Overall, this Monte Carlo simulation revealed that the ISGM performs well under a range of conditions, but also clearly indicated conditions under which estimation problems become more likely with this model. Non-convergence occurred mostly in extreme conditions, that is, when consistency, sample size, and the number of time points were low (growth models are not typically used when consistency is as low as in the current study, so this condition is of relatively little practical concern). Non-convergence was not an issue when the level of consistency was high, the sample size was at least 300, and measurements were available for at least 4 time points. Solutions with an improper residual covariance matrix (Θ) were uncommon (less than 0.2% overall). Replications with an improper latent variable covariance matrix occurred much more frequently in general, and particularly when population correlations were close to 1.0. The theoretical χ^2 distribution was generally well-approximated, with average difference between expected and observed proportions below

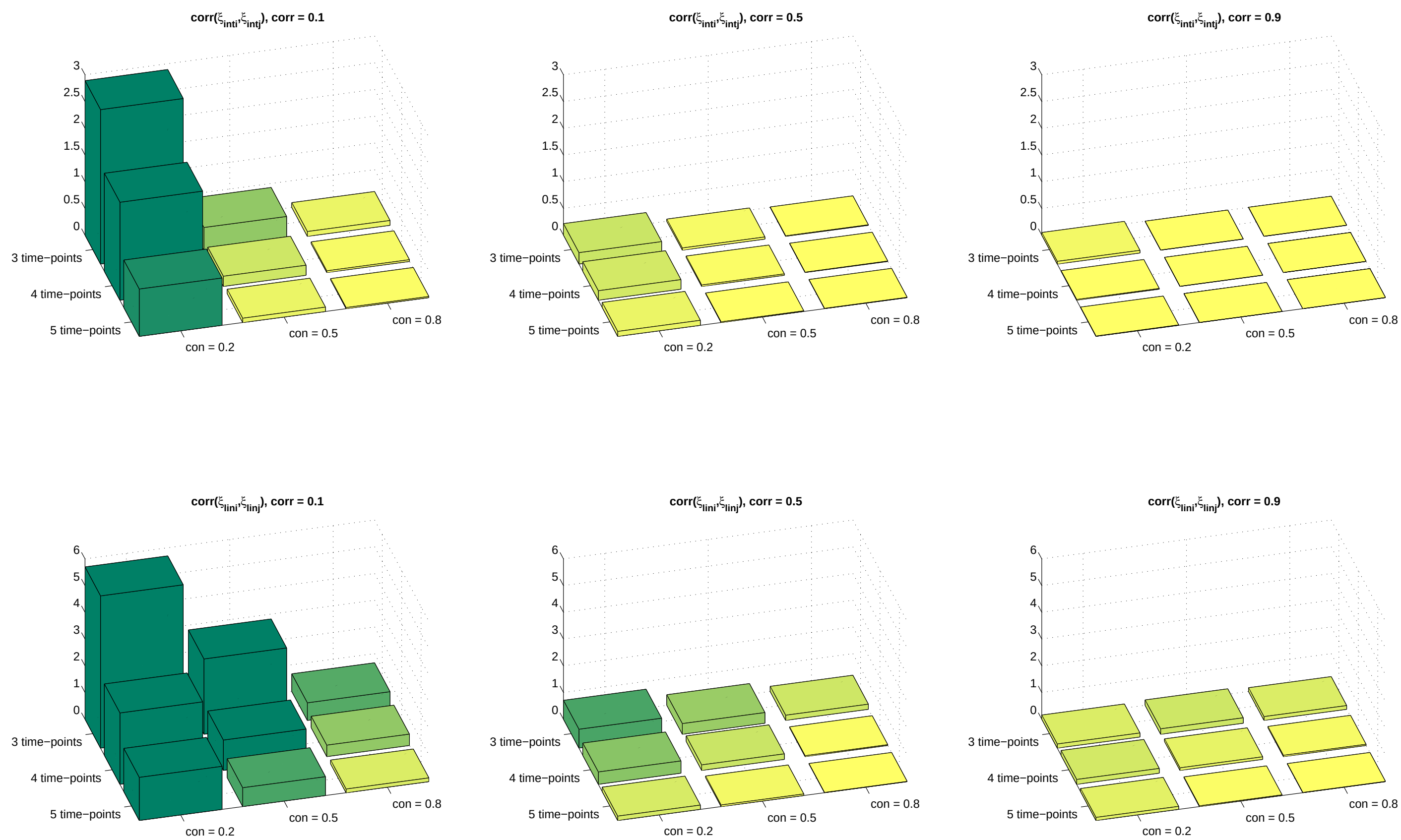


Figure H7. Parameter estimate bias for intercept correlations (top row), and slope correlations (bottom row).

.10 in all conditions. Interestingly, the size of the design had a negative effect on the χ^2 distribution approximation. This is in line with a study by Kenny and McCoach (2003), who also found that CFI and TLI also tend to worsen with the size of design, while RMSEA tends to improve. Most model parameters were accurately reproduced (see Figure H4A), and the corresponding standard error estimates for these model parameters were reliable (see Figure H8). Therefore, both estimated values and inferences made based on these parameter estimates are generally trustworthy, except for the conditions described below.

Three sets of parameters and the corresponding standard errors were not always accurately estimated. These include growth variances, intercept covariances, and slope covariances. Estimates of growth factor variances and the accompanying standard errors were most biased when the level of consistency, number of time points, and sample size was low. These are the same conditions that led to more frequent model non-convergence.

Parameter estimate bias and standard error bias for intercept and slope correlations

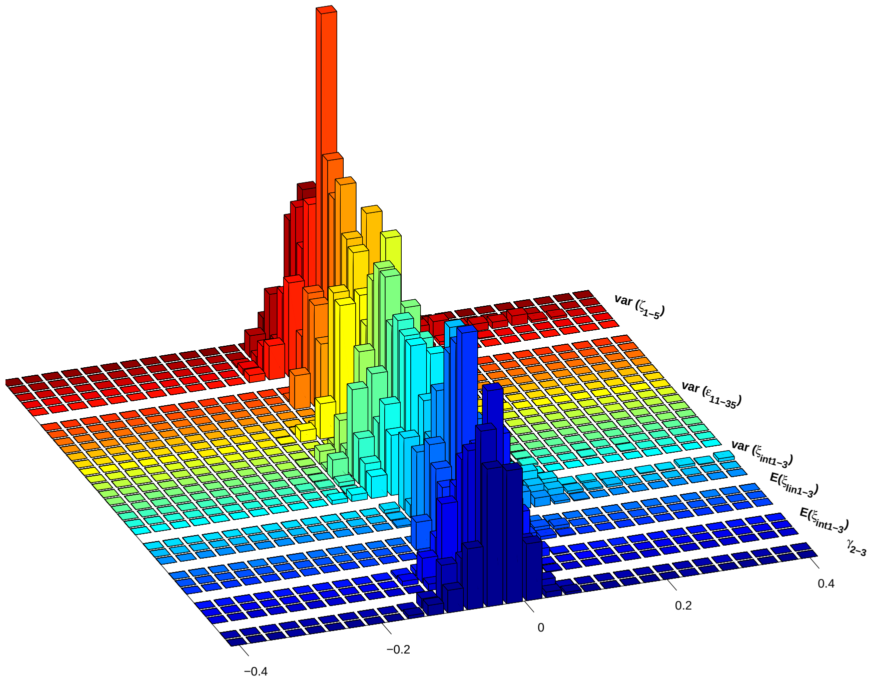


Figure H8. Standard Error Bias. The parameters (front to back) are: γ_2, γ_3, $E(\xi_{\text{int}_1})$, $E(\xi_{\text{int}_2})$, $E(\xi_{\text{int}_3})$, $E(\xi_{\text{lin}_1})$, $E(\xi_{\text{lin}_2})$, $E(\xi_{\text{lin}_3})$, $\text{var}(\xi_{\text{int}_1})$, $\text{var}(\xi_{\text{int}_2})$, $\text{var}(\xi_{\text{int}_3})$, $\text{var}(\varepsilon_{11})$, $\text{var}(\varepsilon_{21})$, $\text{var}(\varepsilon_{31})$, $\text{var}(\varepsilon_{12})$, $\text{var}(\varepsilon_{22})$, $\text{var}(\varepsilon_{32})$, $\text{var}(\varepsilon_{13})$, $\text{var}(\varepsilon_{23})$, $\text{var}(\varepsilon_{33})$, $\text{var}(\varepsilon_{14})$, $\text{var}(\varepsilon_{24})$, $\text{var}(\varepsilon_{34})$, $\text{var}(\varepsilon_{15})$, $\text{var}(\varepsilon_{25})$, $\text{var}(\varepsilon_{35})$, $\text{var}(\zeta_1)$, $\text{var}(\zeta_2)$, $\text{var}(\zeta_3)$, $\text{var}(\zeta_4)$, and $\text{var}(\zeta_5)$.

were high when consistency, correlation, and number of time points were low. Standard error bias was particularly high for slope correlations except under the condition with the highest consistency, correlation, and number of time points. Coverage for the correlation estimates was the highest (average = .935) when the correlation was in the middle range (0.5). Coverage for all parameters except the intercept and slope correlations was very close to the nominal value of .95 (within 1.3%) under all simulation conditions.

In summary, in this simulation we assessed a range of conditions, including extreme conditions of low sample size, low consistency, and high and low latent correlations. This was done in order to give the model a "chance to fail" and to clearly identify conditions under which estimation problems should be expected to occur. Our results indicate that the ISGM shows problems in small samples ($N \leq 200$), which is not unexpected for a complex growth model. We therefore recommend that larger samples ($N = 300$ or larger) be used for this model. We also found that situations with low consistency (i.e., weakly defined growth factors) can be problematic, especially when other conditions are suboptimal as well. Low consistency means that most of the true score variance is due to state variability processes (situational effects or person by situation interaction effects). We suspect that in these cases, the empirical identification of growth factors becomes more difficult, leading to estimation problems. In such cases, a researcher could consider simpler models (e.g. LST models without growth components) that focus exclusively on state-variability

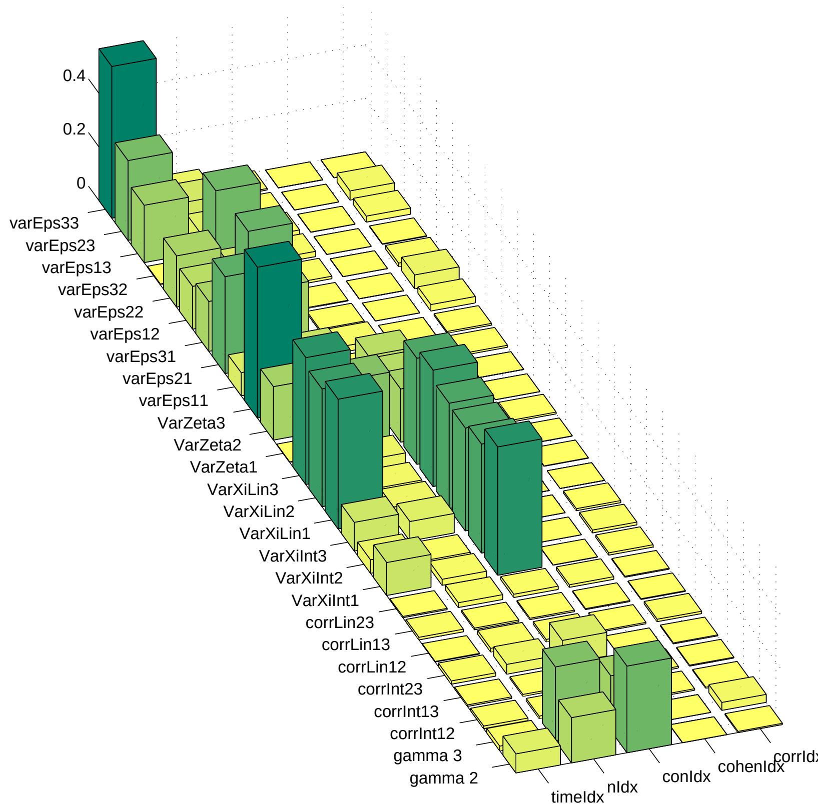


Figure H9. R^2 values for regression of standard error bias by time, N, cohen, and corr. Parameters not estimated in all conditions [e.g., var (ε_{14-35}), var (ζ_{4-5})] were omitted.

processes. Another problematic condition (in terms of improper solutions) was the size of the correlation between growth factors. The size of these correlations indicates the degree of homogeneity of the indicators. Very high correlations (.9) indicate that it is hard to distinguish between the trait components of different indicators, showing that these indicators are essentially homogeneous. In these cases, researchers may consider simplifying the model by choosing a model with general (as opposed to indicator-specific) growth factors (i.e., the SGM or GSGM instead of the ISGM).

The ISGM appears to be most suitable when the level of trait (or growth) variance is substantial relative to occasion-specific (state residual) variability and when trait/growth correlations are substantial, but not too high. This makes sense, as this model is designed to model trait-change processes with related, yet heterogeneous indicators. Another interesting finding of this simulation study was the positive effect of the number of time points on various measures of model performance. Our results showed that the common recommendation according to which researchers should use at least four (and preferably more) time points when modeling growth was confirmed for the ISGM. Having a larger number of time points clearly helped the model to perform better in otherwise suboptimal conditions. We therefore recommend that researchers using the ISGM collect data on at least four—and preferably more—measurement occasions.

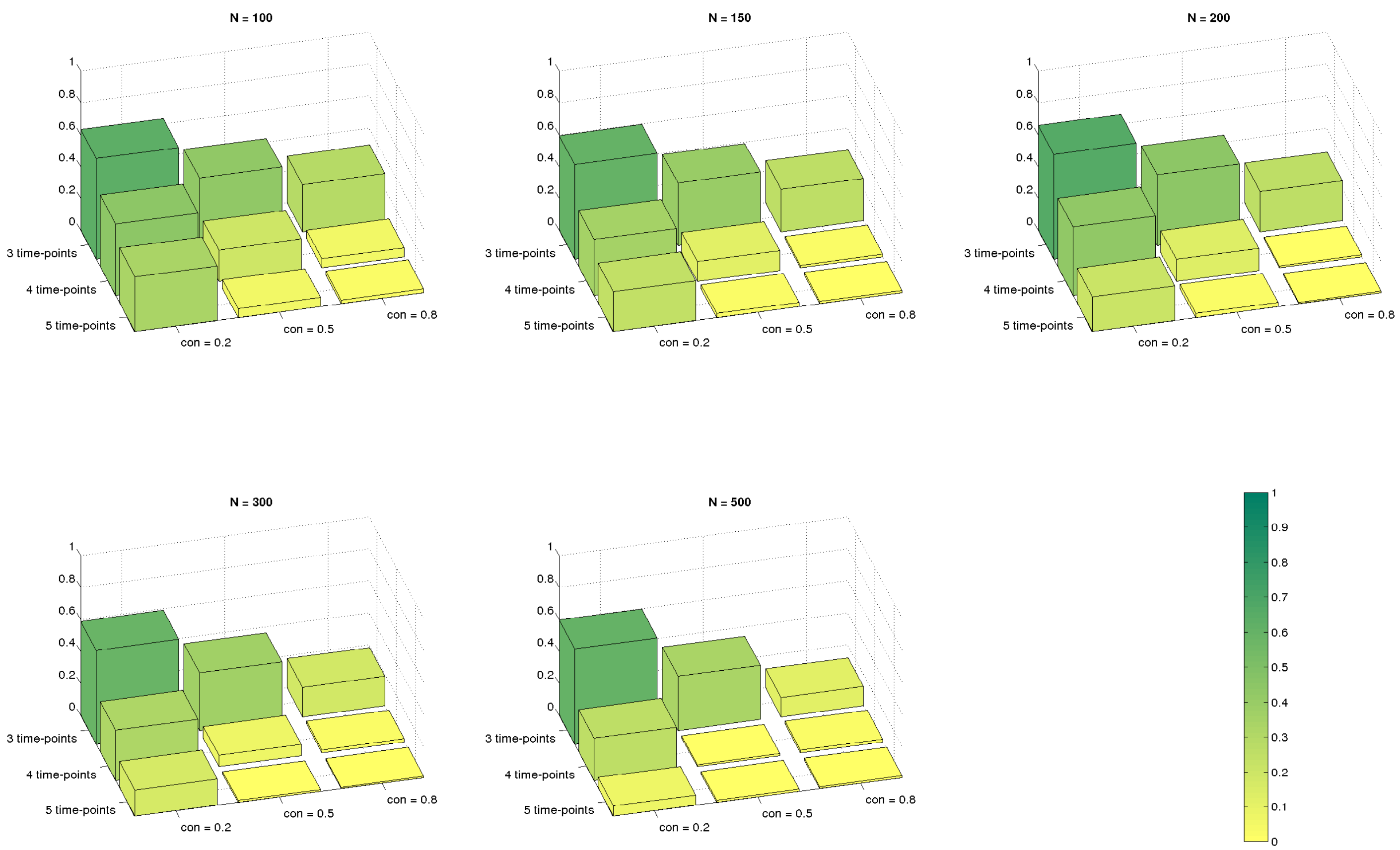


Figure H10. Standard error bias for growth factor variances, var $(\xi_{\text{lin}_{1-3}})$.

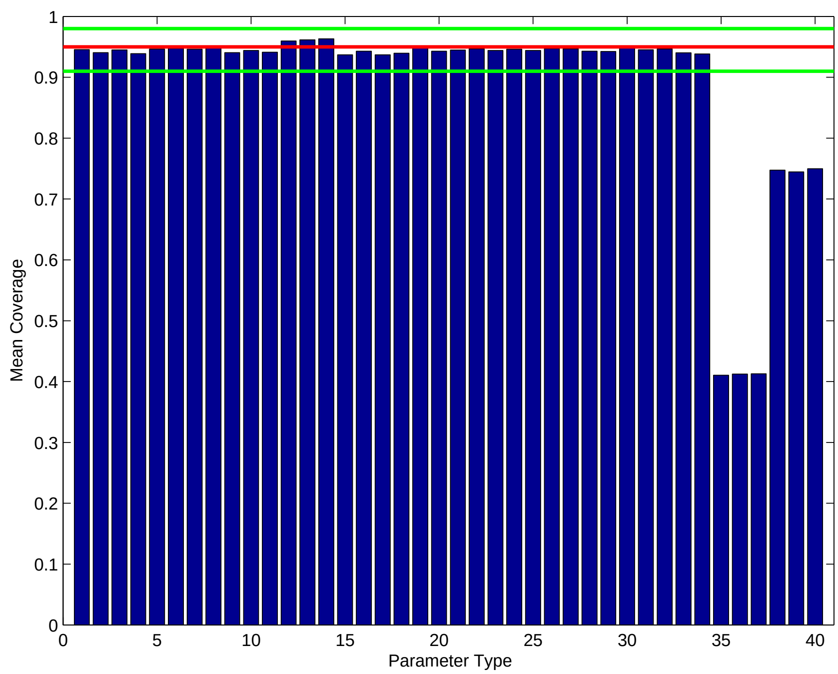


Figure H11. Mean Coverage. The parameters (left to right) are: γ_2, γ_3, $E(\xi_{\text{int}_1})$, $E(\xi_{\text{int}_2})$, $E(\xi_{\text{int}_3})$, $E(\xi_{\text{lin}_1})$, $E(\xi_{\text{lin}_2})$, $E(\xi_{\text{lin}_3})$, $\text{var}(\xi_{\text{int}_1})$, $\text{var}(\xi_{\text{int}_2})$, $\text{var}(\xi_{\text{int}_3})$, $\text{var}(\xi_{\text{lin}_1})$, $\text{var}(\xi_{\text{lin}_2})$, $\text{var}(\xi_{\text{lin}_3})$, $\text{var}(\zeta_1)$, $\text{var}(\zeta_2)$, $\text{var}(\zeta_3)$, $\text{var}(\zeta_4)$, $\text{var}(\zeta_5)$, $\text{var}(\varepsilon_{11})$, $\text{var}(\varepsilon_{21})$, $\text{var}(\varepsilon_{31})$, $\text{var}(\varepsilon_{12})$, $\text{var}(\varepsilon_{22})$, $\text{var}(\varepsilon_{32})$, $\text{var}(\varepsilon_{13})$, $\text{var}(\varepsilon_{23})$, $\text{var}(\varepsilon_{33})$, $\text{var}(\varepsilon_{14})$, $\text{var}(\varepsilon_{24})$, $\text{var}(\varepsilon_{34})$, $\text{var}(\varepsilon_{15})$, $\text{var}(\varepsilon_{25})$, $\text{var}(\varepsilon_{35})$, $\text{corr}(\xi_{\text{int}_1}, \xi_{\text{int}_2})$, $\text{corr}(\xi_{\text{int}_1}, \xi_{\text{int}_3})$, $\text{corr}(\xi_{\text{lin}_1}, \xi_{\text{lin}_2})$, $\text{corr}(\xi_{\text{lin}_1}, \xi_{\text{lin}_3})$, and $\text{corr}(\xi_{\text{lin}_2}, \xi_{\text{lin}_3})$. The red line is drawn at the nominal .95 level. The green lines are drawn at .91 and .98.

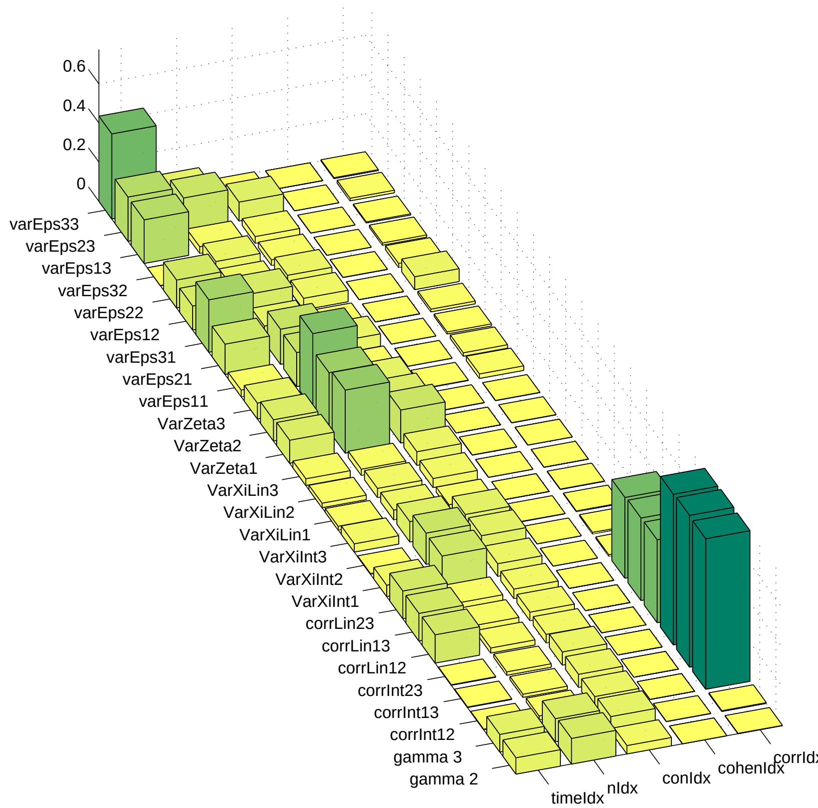


Figure H12. R^2 values for regression of coverage on time, N, cohen, and corr. Parameters not estimated in all conditions [e.g., var (ε_{14-35}), var (ζ_{4-5})] were omitted.

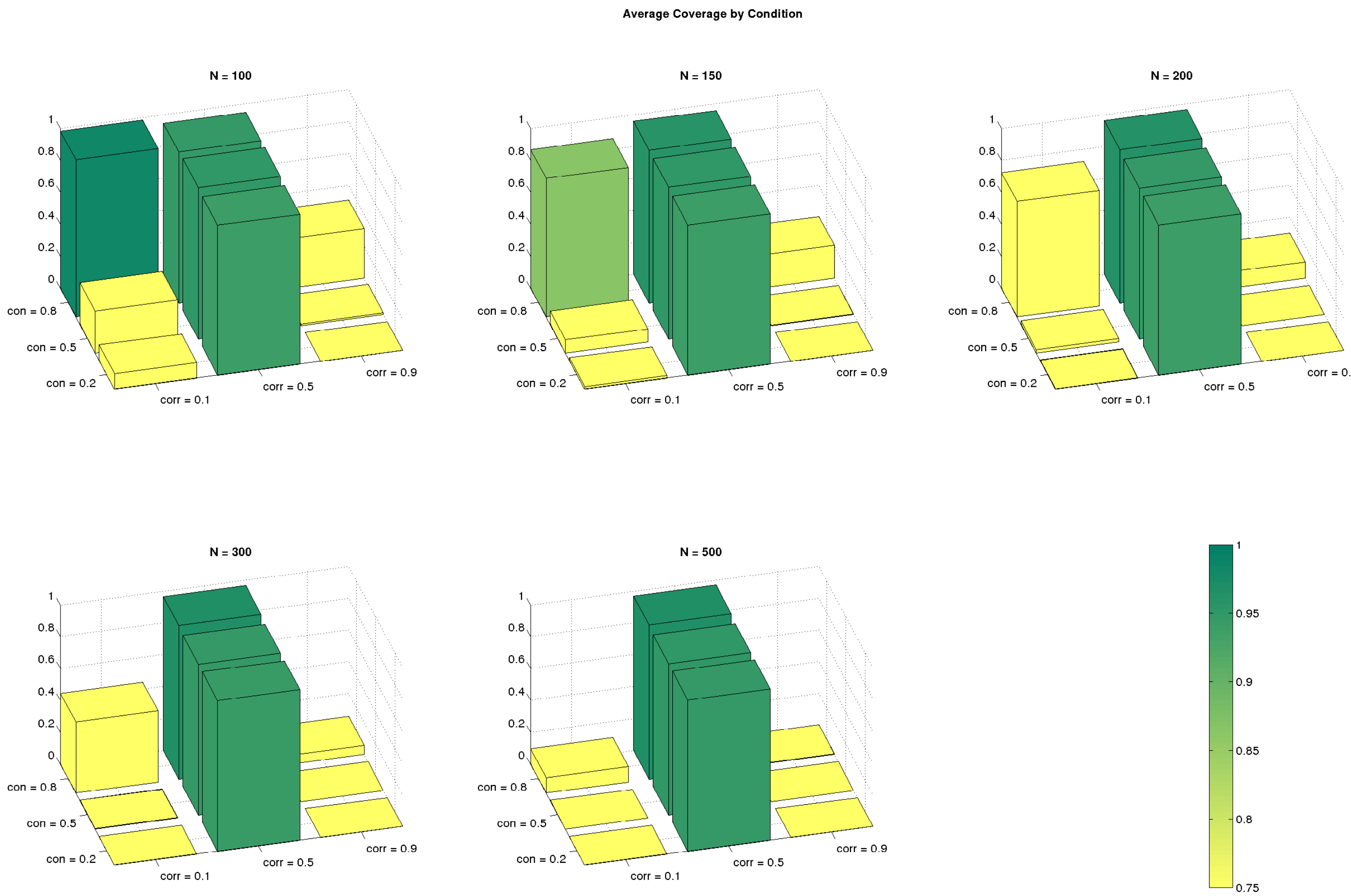


Figure H13. Coverage for corr $\left(\xi_{\text{int}_i}, \xi_{\text{int}_j}\right)$.

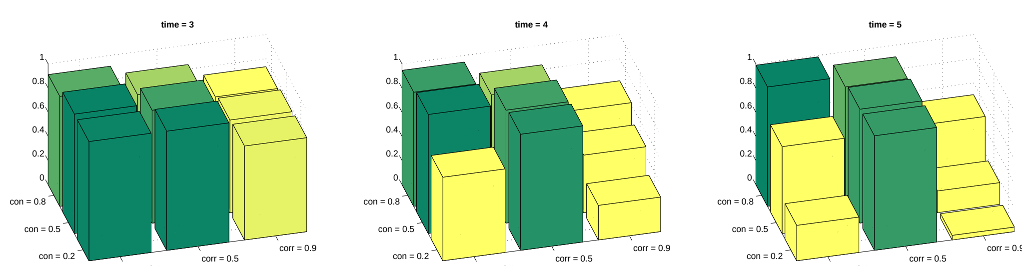


Figure H14. Coverage for corr $\left(\xi_{\text{lin}_i}, \xi_{\text{lin}_j}\right)$.

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